

## Day 4 (ct'd)

### Learning Goals:

- Define categories
- Unify sets, vector spaces, top'l spaces, ... in this language

Defn A category  $C$  consists of

- a class of objects  $ob C$ ,
- for  $X, Y \in ob C$ , a set  $C(X, Y)$  of morphisms with domain  $X$  and codomain  $Y$  with elts denoted  $f: X \rightarrow Y$ ,
- a composition rule assigning  $gf: X \rightarrow Z$  to  $f: X \rightarrow Y, g: Y \rightarrow Z$ , i.e.,  
 $C(Y, Z) \times C(X, Y) \rightarrow C(X, Z)$

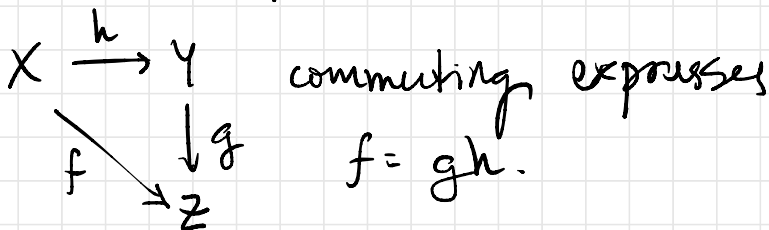
satisfying two conditions:

(i) (associativity) for  $X \xrightarrow{h} Y \xrightarrow{g} Z \xrightarrow{f} W$ ,  
 $f(gh) = (fg)h$ ,

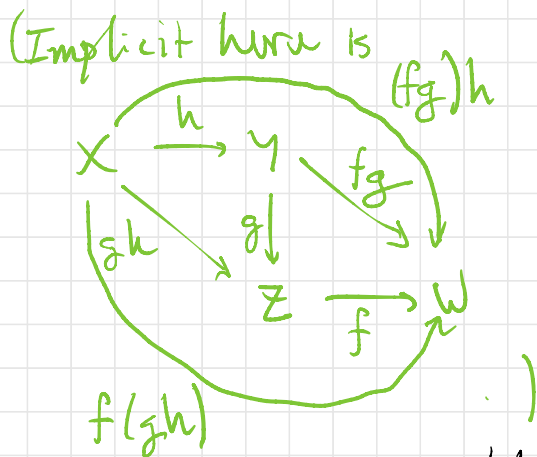
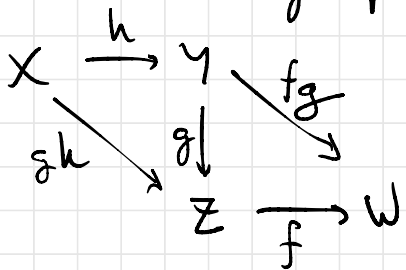
(ii) (identity)  $\forall X \in ob C \exists id_X: X \rightarrow X$  s.t.  
 $f id_X = f = id_Y f$ .

Moral ex: Identity morphisms are unique.

Often use commutative diagrams to express composition in a category and convey equality of various compositions:



The associativity of composition is equivalent to commutativity of



E.g. (1) Set: sets + functions

w/ composition =

composition of funcs

Objects equal to set of sets??

Following tradition, we will be fast and loose with "size issues."

(2)  $\text{Fin}$ : finite sets + functions

(3)  $\text{Set}_*$  and  $\text{Fin}_*$ : pointed (finite) sets + pointed functions

A pointed set is a pair  $(A, a)$  with  $a \in A$  and a pointed function  $f: (A, a) \rightarrow (B, b)$  is a function  $f: A \rightarrow B$  s.t.  $f(a) = b$ .

(4)  $\text{Top}$ : topological spaces + continuous functions

(5)  $\text{Top}_*$ : pointed top'l spaces + ptd cts fns.

(6) Fix a field  $k$ .  $\text{Vect}_k$  and  $\text{FinVect}_k$ :

(finite-dimensional)  $k$ -vector spaces + linear transformations.

(7) For  $G$  a group (or monoid) get an associated category  $\mathcal{B}G$  with  $\text{ob } \mathcal{B}G = \{ \bullet \}$ ,  $\mathcal{B}G(\bullet, \bullet) = G$ , composition given by group (monoid) law.

(8)  $\text{Grp}$ ,  $\text{AbGrp}$ ,  $\text{Ring}$ ,  $\text{CRing}$ ,  $\text{Alg}_k, \dots$ :

algebraic structure + homomorphisms

(9) For  $(P, \leq)$  a partially ordered set, have

associated category (also denoted  $\mathcal{P}$ ) with objects  $\mathcal{P}$  and  $\mathcal{P}(a,b) = \begin{cases} \emptyset & \text{if } a \not\leq b \\ * & \text{if } a \leq b \end{cases}$

generic notation for singleton set.

(10) For any category  $\mathcal{C}$  have its opposite category  $\mathcal{C}^{\text{op}}$  with  $\text{ob } \mathcal{C}^{\text{op}} = \text{ob } \mathcal{C}$ ,  $\mathcal{C}^{\text{op}}(X,Y) = \mathcal{C}(Y,X)$  and inherited composition.

TPS Other examples?

Now address invertibility of morphisms and "sameness" within a category.

Defn For  $f: X \rightarrow Y$  a morphism in a category  $\mathcal{C}$ ,

(i)  $f$  is left invertible when  $\exists g: Y \rightarrow X$  s.t.

$gf = \text{id}_X$ ;  $g$  is a left inverse of  $f$

$$\left( \begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \text{id}_X & \downarrow g \\ & & X \end{array} \right)$$

(ii)  $f$  is right invertible when  $\exists h: Y \rightarrow X$

s.t.  $fh = \text{id}_Y$ ;  $h$  is a **right inverse** of  $f$

$$\left( \begin{array}{ccc} & Y & \\ h \downarrow & \searrow \text{id}_Y & \\ X & \xrightarrow{f} & Y \end{array} \right)$$

When  $f$  has a left inverse  $g$  and right inverse  $h$ , then  $g=h$  is the (**two-sided**) inverse of  $f$ . So

(iii)  $f$  is **invertible** and is called an **isomorphism** when it is both left and right invertible.

$$\left( \text{id}_X \circ X \begin{array}{c} \xleftarrow{g} \\ \xrightarrow{f} \\ \end{array} Y \circ \text{id}_Y \right)$$

Call objects  $X, Y$  **isomorphic** (in  $\mathcal{C}$ ) when there exists an isomorphism  $f: X \rightarrow Y$ ; write  $X \cong Y$ .

- E.g.**
- Isomorphisms (isos) in Set are bijections.
  - Isos in Top are **homeomorphisms**.
  - If every morphism in  $\mathcal{C}$  is an iso,

call  $C$  a **groupoid**. The category  $iC$  w/ same objects as  $C$  and only  $i$  as morphisms is the **maximal groupoid** of  $C$ .

Defn **Topology** is the study of **topological properties**, i.e., properties preserved by homeomorphism.

Eg. • Cardinality is a topological property since every homeomorphism is a bijection.

◇  $\exists$  cts bij's that are not homeomorphisms

- Less obvious/not yet defined: metrizability, connectedness, compactness, Hausdorffness, ... are top'l properties.
- Completeness and boundedness of metric spaces are not top'l properties since

$$\begin{array}{ccc} (-1, 1) & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \frac{x}{1-x^2} \end{array} \quad \text{is a homeo.}$$