Day 4 (ct'd)

Learning Goals:

- Define categories
- Unify sets, vector spaces, top'l spaces, ... in this language

Defn A **category** C consists of

(i) a class of **objects** $\mathrm{ob}\,C$,

(ii) for $X, Y \in \mathrm{ob}\,C$, a set $C(X,Y)$ of **morphisms** with **domain** X and **codomain** Y with elts denoted $f: X \to Y$,

(iii) a **composition rule** assigning $gf: X \to Z$ to $f: X \to Y$, $g: Y \to Z$, i.e.,

$$C(Y,Z) \times C(X,Y) \longrightarrow C(X,Z)$$

satisfying two conditions:

(i) (**associativity**) for $X \xrightarrow{h} Y \xrightarrow{g} Z \xrightarrow{f} W$,

$$f(gh) = (fg)h,$$

(ii) (**identity**) $\forall X \in \mathrm{ob}\,C \;\exists\, \mathrm{id}_X : X \to X$ s.t.

$$f\,\mathrm{id}_X = f = \mathrm{id}_Y f.$$

Moral exc: Identity morphisms are unique.

Often use commutative diagrams to express composition in a category and convey equality of various compositions:

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

commuting expresses $f = gh$.

The associativity of composition is equivalent to commutativity of

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "gh"'] & Y \arrow[d, "g"'] \arrow[dr, "fg"] & \\ & Z \arrow[r, "f"'] & W \end{tikzcd}$$

(Implicit here is

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "gh"'] \arrow[drr, bend left=60, "(fg)h"] \arrow[drr, bend right=60, "f(gh)"'] & Y \arrow[d, "g"'] \arrow[dr, "fg"] & \\ & Z \arrow[r, "f"'] & W \end{tikzcd}$$

)

E.g. (1) **Set**: sets + functions w/ composition = composition of func'ns

Object equal to set of sets ?? Following tradition, we will be fast and loose with "size issues."

(2) Fin : finite sets + functions

(3) Set_* and Fin_* : pointed (finite) sets + pointed functions

A pointed set is a pair (A, a) with $a \in A$ and a pointed function $f : (A, a) \longrightarrow (B, b)$ is a function $f : A \to B$ s.t. $f(a) = b$.

(4) Top : topological spaces + continuous functions

(5) Top_* : pointed top'l spaces + ptd cts fns.

(6) Fix a field k. Vect_k and $\mathrm{FinVect}_k$: (finite-dimensional) k-vector spaces + linear transformations.

(7) For G a group (or monoid) get an associated category BG with $\mathrm{ob}\, BG = \{\bullet\}$, $BG(\bullet, \bullet) = G$, composition given by group (monoid) law.

(8) Gp, AbGp, Ring, CRing, Alg_k, ... : algebraic structure + homomorphisms

(9) For $(P, \leq)$ a partially ordered set, have

associated category (also denoted P) with objects P and $P(a,b) = \begin{cases} \emptyset & \text{if } a \nleq b \\ * & \text{if } a \leq b \end{cases}$

generic notation for singleton set

(10) For any category C have its opposite category C^{op} with $\mathrm{ob}\, C^{op} = \mathrm{ob}\, C$, $C^{op}(X,Y) = C(Y,X)$ and inherited composition.

TPS Other examples?

Now address invertibility of morphisms and "sameness" within a category.

Defn For $f : X \to Y$ a morphism in a category C,

(i) f is left invertible when $\exists g : Y \to X$ s.t. $gf = id_X$; g is a left inverse of f

$$\left(\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "id_X"'] & Y \arrow[d, "g"] \\ & X \end{tikzcd}\right)$$

(ii) f is right invertible when $\exists h : Y \to X$

s.t. $fh = \mathrm{id}_Y$; h is a right inverse of f

$$\left(\begin{tikzcd} Y \arrow[d, "h"'] \arrow[dr, "\mathrm{id}_Y"] & \\ X \arrow[r, "f"'] & Y \end{tikzcd}\right)$$

When f has a left inverse g and right inverse h, then $g = h$ is <u>the</u> (two-sided) inverse of f. So

(iii) f is invertible and is called an isomorphism when it is both left and right invertible.

$$\left(\begin{tikzcd} X \arrow[loop left, "\mathrm{id}_X"] \arrow[r, bend right, "f"'] & Y \arrow[l, bend right, "g"'] \arrow[loop right, "\mathrm{id}_Y"] \end{tikzcd}\right)$$

Call objects X, Y isomorphic (in C) when there exists an isomorphism $f : X \longrightarrow Y$; write $X \cong Y$.

E.g.
- Isomorphisms (isos) in Set are bijections.
- Isos in Top are homeomorphisms.
- If every morphism in C is an iso,

call C a groupoid. The category iC w/ same objects as C and only isos as morphisms is the maximal groupoid of C.

Defn Topology is the study of topological properties, i.e., properties preserved by homeomorphism.

E.g.

- Cardinality is a topological property since every homeomorphism is a bijection.

  $\not\Leftarrow$ $\exists$ cts bij'ns that are not homeomorphisms
- Less obvious/not yet defined: metrizability, connectedness, compactness, Hausdorffness, ... are top'l properties.
- Completeness and boundedness of metric spaces are not top'l properties since

$$(-1,1) \longrightarrow \mathbb{R} \quad \text{is a homeo.}$$
$$x \longmapsto \frac{x}{1-x^2}$$