Day 3 Learning goals:

- Clarify "coarsest topology containing..."
- Metric spaces and the metric topology.
- Continuity

Mea culpa Day 2 failed to justify the existence of a coarsest topology containg $\mathcal{B}$.

We can actually make this construction for any subset $S \subseteq 2^X$; call the corresponding topology $\langle S \rangle$.

Lemma The (finite or infinite) intersection of topologies is a topology.

Pf Moral exercise — see #remote. $\square$

Prop If $S \subseteq 2^X$, then
$$\bigcap_{\substack{T \supseteq S \\ T \text{ a topology}}} T = \langle S \rangle,$$
i.e. $\langle S \rangle$ exists and is equal to the given int'n of topologies.

Pf The discrete topology 2^X contains S, so we are intersecting over a nonempty collection of topologies. By the lemma, $\bigcap_{\substack{T \supseteq S \\ T \text{ top}}} T$ is a topology on X.

It is the coarsest such topology since $S \subseteq T'$ a topology $\Rightarrow \bigcap T \subseteq T'$. $\square$

Metric spaces

Defn A **metric space** is a pair (X, d), X a set, $d: X \times X \longrightarrow \mathbb{R}$ satisfying

- (nonnegativity) $d(x, y) \geq 0$
- (symmetry) $d(x, y) = d(y, x)$
- (Δ inequality) $d(x, z) \leq d(x, y) + d(y, z)$
- (identity of indiscernibles) $d(x, y) = 0 \Leftrightarrow x = y$

(All properties universally quantified.)

e.g. • $X = \mathbb{R}^n$, $p \geq 1$, $x, y \in \mathbb{R}^n$

$$d_p(x, y) = \left(\sum_{i=1}^{n} (y_i - x_i)^p \right)^{1/p}$$

$$d_\infty(x, y) = \sup_{1 \leq i \leq n} |y_i - x_i|$$

For $p = 2$, recover Euclidean distance.

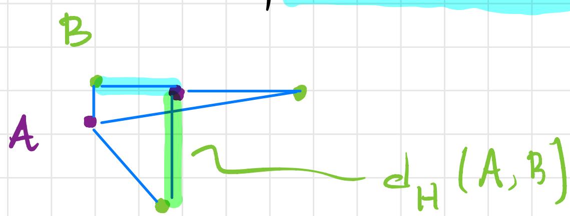
- For any X, have discrete metric

$$d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}.$$

- Suppose (X,d) is a metric space and let $\mathrm{Ran}(X) = \{\text{finite subsets of } X\}$.

  For $A, B \in \mathrm{Ran}(X)$, define the Hausdorff distance

$$d_H(A,B) = \max\left\{ \underbrace{\max_{a\in A} \min_{b\in B} d(a,b)}, \underbrace{\max_{b\in B} \min_{a\in A} d(a,b)} \right\}.$$



$$= \inf\{\varepsilon > 0 \mid A \subseteq \underbrace{B_\varepsilon}_{=\{x\in X \,\mid\, d(x,b)<\varepsilon \text{ for some } b\in B\}},\ B \subseteq A_\varepsilon\}$$

<u>Defn</u> If (X,d) a metric space, $x \in X$, and $r > 0$ then the ball of radius r centered at x is

$$B(x,r) := \{ y \in X \mid d(x,y) < r \}.$$

The metric basis for X is

$$\mathcal{B}_d = \{ B(x,r) \mid x \in X,\ r > 0 \}.$$

The metric topology on X is

$$\langle \mathcal{B} \rangle = \{ U \subseteq X \mid U \text{ is a union of balls} \}.$$

A topological space $(X, \mathcal{T})$ is metrizable when $\exists$ metric d such that $\mathcal{T} = \langle \mathcal{B}_d \rangle$.

Multiple metrics induce the same topology.

E.g. $$\bar{d}(x,y) = \begin{cases} d(x,y) & \text{if } d(x,y) < 1 \\ 1 & \text{o/w}. \end{cases}$$

satisfied $\langle \mathcal{B}_d \rangle = \langle \mathcal{B}_{\bar{d}} \rangle$ for all metrics d.

Also $\langle \mathcal{B}_{d_p} \rangle = \langle \mathcal{B}_{d_q} \rangle$ for all $1 \le p, q \le \infty$.

(See HW.)

Prop If $Y \subseteq X$ and (X,d) is a metric space then $d|_{Y \times Y} : Y \times Y \to \mathbb{R}$ is a metric on Y. $\square$

e.g. $I = [0,1] \subseteq \mathbb{R}$, $D^n = \{x \in \mathbb{R}^n \mid x \cdot x \leq 1\} \subseteq \mathbb{R}^n$,

$S^n = \{x \in \mathbb{R}^{n+1} \mid x \cdot x = 1\} \subseteq \mathbb{R}^{n+1}$ are all metric (and topological) spaces in this fashion.

Defn A function $f : X \longrightarrow Y$ between top'l spaces is continuous when $U \subseteq Y$ open $\Rightarrow f^{-1}U \subseteq X$ open.

preimage $= \{x \in X \mid f(x) \in U\}$

Prop Suppose the topologies on X, Y are gen'd by bases $\mathcal{B}, \mathcal{C}$, resp. Then $f : X \longrightarrow Y$ continuous iff $\forall y \in Y$ and $x \in X$ s.t. $f(x) = y$, if $y \in C \in \mathcal{C}$ then $\exists\, x \in B \in \mathcal{B}$ s.t. $fB \subseteq C$.

Cor A function between metric spaces $f : X \longrightarrow Y$ is continuous iff $\forall y \in Y$ and $x \in X$ s.t. $f(x) = y$, $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $f B(x,\delta) \subseteq B(y,\varepsilon)$. $\square$

Pf Prop Suppose f is cts, $f(x) = y$, and $y \in C \in \mathcal{C}$. Then $f^{-1}C$ is open so $x \in f^{-1}C = \bigcup_{B \in I} B$

for some $\mathcal{I} \subseteq \mathcal{B}$. Must have x in one of these B, and $B \subseteq f^{-1}C \Rightarrow fB \subseteq C$. ✓

Converse: HW (immoral exc?) $\square$

Prop Given topological spaces X, Y, Z and continuous functions $f: X \to Y$ and $g: Y \to Z$, have that:

- $id_X : X \to X$
- $gf : X \to Z$

are continuous.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "gf"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

we'll write gf for $g \circ f$, the composite function $x \mapsto g(f(x))$.

This makes top'l spaces + continuous functions a category.

person w/ spyglass?

Pf id_X: $\forall U \subseteq X$ open, $id_X^{-1} U = U$ open. ✓

gf: Fix $U \subseteq Z$ open. Moral exc:

$(gf)^{-1} U = f^{-1}[g^{-1} U]$. Since g cts, $g^{-1} U$ open. Since f cts, $f^{-1}[g^{-1} U]$ open. ✓ $\square$