# Day 2

**Learning goals:**

- definition of a topological space
- specifying topologies by basis

**Defn** A *topological space* consists of a set $X$ and a collection of subsets $\mathcal{T} \subseteq 2^X$ (open sets) satisfying the following:

(i) $\emptyset, X \in \mathcal{T}$

(ii) any union of elements of $\mathcal{T}$ remains in $\mathcal{T}$ (potentially infinite!)

(iii) any finite intersection of elements of $\mathcal{T}$ remains in $\mathcal{T}$.

**Slogan:** Open sets are closed under arbitrary union and finite intersection.

**Defn** $V \subseteq X$ is closed when $X \setminus V \in \mathcal{T}$, i.e. $X \setminus V$ is open.

Linguistic atrocity: $\emptyset, X$ are clopen in every topology on $X$.

We will live with this formalism and develop several examples before attempting to justify this definition.

Bourbaki examples

- $(X, 2^X)$ is the discrete topology on $X$
- $(X, \{\emptyset, X\})$ is the indiscrete topology on $X$ (aka codiscrete, trivial, or chaotic topology).

Analysis example

- $X = \mathbb{R}$, $\quad \mathcal{T} = \left\{ U \subseteq \mathbb{R} \;\middle|\; \begin{array}{l} \forall x \in U \;\; \exists a < b \in \mathbb{R} \\ \text{s.t. } (a,b) \subseteq U \end{array} \right\}$

$$= \left\{ U \subseteq \mathbb{R} \;\middle|\; \begin{array}{l} U \text{ is a union of} \\ \text{open intervals} \end{array} \right\}$$

This is the standard topology on $\mathbb{R}$.

We'll see more examples soon.

Suppose $\mathcal{T}, \mathcal{T}'$ are topologies on $X$ and $\mathcal{T} \subseteq \mathcal{T}'$. We call $\mathcal{T}$ coarser than $\mathcal{T}'$ and $\mathcal{T}'$ finer than $\mathcal{T}$. We will often construct topologies as the "coarsest" (or "smallest") topology satisfying a particular property.

One such example is when we generate a topology with a basis:

**Defn** A collection $\mathcal{B} \subseteq 2^X$ is a basis for a topology on $X$ when

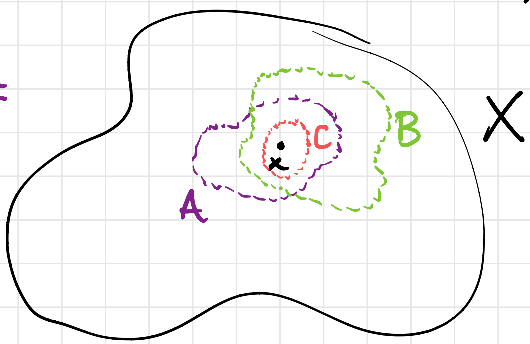
i.e. $\bigcup_{B \in \mathcal{B}} B = X$

(i) $\forall x \in X \ \exists B \in \mathcal{B}$ s.t. $x \in B$

(ii) $x \in A \cap B$, $A, B \in \mathcal{B} \implies \exists C \in \mathcal{B}$ s.t. $x \in C \subseteq A \cap B$

every pairwise intersection of basis sets is a union of basis sets

**Cartoon**




The topology $\mathcal{T} = \langle \mathcal{B} \rangle$ generated by $\mathcal{B}$

is the coarsest topology containing $\mathcal{B}$.

**Prop** If $\mathcal{B}$ is a basis for a topology on $X$, then
$$\langle \mathcal{B} \rangle = \left\{ U \subseteq X \;\middle|\; \begin{array}{l} \forall x \in U \ \exists B \in \mathcal{B} \\ \text{s.t. } x \in B \subseteq U \end{array} \right\}$$

"Obvious"?? Make sure you could prove this if compelled.

$$= \left\{ U \subseteq X \;\middle|\; \begin{array}{l} U \text{ is a union of} \\ \text{elements of } \mathcal{B} \end{array} \right\}.$$

**Pf** Write $T'$ for the set of unions of sets in $\mathcal{B}$. Since opens are closed under unions, $T' \subseteq \langle \mathcal{B} \rangle$. Since $B \in \mathcal{B}$ is the "one set union" of itself, $\mathcal{B} \subseteq T'$. Thus it suffices to show that $T'$ is a topology (because $\langle \mathcal{B} \rangle$ is the coarsest topology containing $\mathcal{B}$, so $\mathcal{B} \subseteq T' \Rightarrow \langle \mathcal{B} \rangle \subseteq T'$ if $T'$ is a topology!).

A "union of unions is a union" so $T'$ is closed under unions.

Closure under finite intersections is equivalent

to closure under pairwise intersection
(induction + associativity of $\cap$)

so suffices to show $U, V \in \mathcal{T}' \Rightarrow U \cap V \in \mathcal{T}'$.

If $U, V \in \mathcal{T}'$, then we may write

$$U = \bigcup_{B \in I \subseteq \mathcal{B}} B \quad , \quad V = \bigcup_{C \in J \subseteq \mathcal{B}} C$$

so $$U \cap V = \bigcup_{\substack{B \in I \\ C \in J}} B \cap C.$$ (moral exercise: check this!)

But each $B \cap C$ is a union of elts of $\mathcal{B}$ by (ii), so $U \cap V \in \mathcal{T}'$ as desired. $\square$

NB · $\mathbb{R}^{std}$ has basis $\mathcal{B}^{std} = \{ (a,b) \mid a < b \in \mathbb{R} \}$.

· Next time we will generalize this by taking collections of open balls as a basis for a metric topology.