Topology & Categories

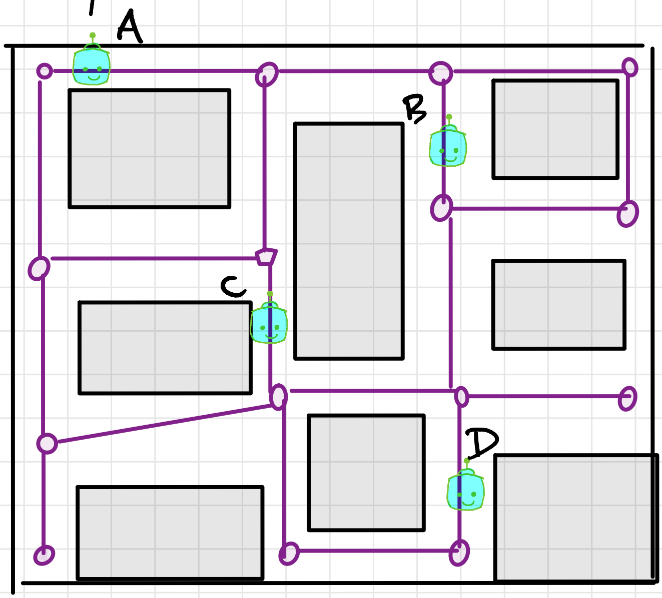
Day 1

Learning Goals:

- Course policies
- Course design
- The big picture

Existence is often a topological matter

Toy example: robots in a warehouse



Question: Given a start and end configuration of A, B, C, D in the warehouse, is there a route the robots the robots can take from start to end without colliding?

Formalization:

X = "spine" of the corridors $\subseteq \mathbb{R}^2$

$\mathrm{Conf}_4(X)$ = configurations of 4 labeled points in X

$= X^4 - \Delta$ "fat" diagonal with 2 or more coords equal

$S, E \in \mathrm{Conf}_4(X)$

Is there a path in $\mathrm{Conf}_4(X)$ from S to E?

continuous function

$$\gamma : [0,1] \longrightarrow \mathrm{Conf}_4(X)$$
$$0 \longmapsto S$$
$$1 \longmapsto E$$

N.B. • "Continuous" ???

• The definition of $\mathrm{Conf}_4(X)$ guarantees no collisions.

Goals: (1) "Topologize" $\mathrm{Conf}_4(X)$ so that small changes in positions of A, B, C, D correspond to nearby "points" in $\mathrm{Conf}_4(X)$.

(2) Generalize "continuity" so that

$$|t-s| \text{ small} \Longrightarrow \gamma(t), \gamma(s) \text{ "close".}$$

(3) Develop tools that actually solve the problem!

Topology is general

A topology on a set gives you a notion of "closeness" but not of distance. The following objects carry important (but potentially non-intuitive) topologies:

- Spec(R), the set of prime ideals in a commutative ring R

  like a field, but don't necessarily have multiplicative inverses, e.g. $\mathbb{Z}$, $\mathbb{Z}/6\mathbb{Z}$, $\mathbb{C}[x,y]$, ...

- Cartesian products and disjoint unions of topological spaces
- Stone spaces (Cantor set, $\mathbb{Z}_p$)
- the set of regular icosahedra inscribed in a fixed sphere

  example of a quotient space

- the set of functions $X \longrightarrow Y$ for X, Y fixed topological spaces.

Topology is a "generous arena" in which we can simultaneously house and analyze these objects.

Relationships and categories

Going back to our robots, suppose we have another corridor layout Y and a continuous function $f: X \longrightarrow Y$. If f is an embedding this induces $\mathrm{Conf}_4(f): \mathrm{Conf}_4(X) \longrightarrow \mathrm{Conf}_4(Y)$

$$(A, B, C, D) \longmapsto (fA, fB, fC, fD)$$

which is also continuous. Moreover, Conf_4 respects the identity function and composition. This makes Conf_4 a functor

$$\mathbf{Emb} \longrightarrow \mathbf{Top}.$$

topological spaces + embeddings

topological spaces + continuous functions

A new categorical imperative:

Relationships take priority over objects

The language of categories, functors, and universal properties will streamline our approach to topological constructions and allow us to draw parallels with other branches of mathematics; it will also result in conceptual, elegant proofs.