MATH 342: TOPOLOGY
HOMEWORK DUE FRIDAY WEEK 13

Problem 1. Recall that a subspace Y of X is called a *retract* of X if there exists a continous map $r\colon X \to Y$ such that $ri = \mathrm{id}_Y$ for $i\colon Y \hookrightarrow X$ the inclusion mapping. Let $\beta\colon \mathsf{Top} \to \mathsf{CH}$ be the Stone-Čech compactification functor. Prove that any compact Hausdorff space X is a retract of βUX for $U\colon \mathsf{CH} \to \mathsf{Top}$ the inclusion of CH as a subcategory of Top. (*Hint*: Think about the counit of the adjunction (β, U); or think about limits of ultrafilters; or think about both.)

Problem 2. Let $[0,1]^{[0,1]}$ denote the set of funtions $[0,1] \to [0,1]$ with the product topology. Part (a) below is an optional challenge problem; you may use its conclusion in (c).

(a) (Challenge — optional.) Prove that $[0,1]^{[0,1]}$ has a countable dense subset and thus is a compactification of $\mathbb{N}$.
(b) Prove that every compactification of a space X is a quotient of βX.
(c) Use (a) and (b) to conclude that the cardinality of $\beta\mathbb{N}$ is at least that of $[0,1]^{[0,1]}$ (*i.e.* that there is a surjective function $\beta\mathbb{N} \to [0,1]^{[0,1]}$).

Problem 3. Find spaces X and Y for which the evaluation map

$$X \times \mathsf{Top}(X, Y) \to Y$$

is not continuous (where $\mathsf{Top}(X, Y)$ has the compact-open topology).