MATH 342: TOPOLOGY
HOMEWORK DUE FRIDAY WEEK 10

Problem 1. Let $f\colon X \to Y$ be a function and suppose that $\mathscr{F}$ is an ultrafilter on X. Prove that the pushforward $f_*\mathscr{F}$ of $\mathscr{F}$ along f is an ultrafilter on Y.[1]

Problem 2. Let $X = \{0,1\}^{[0,1)}$ be the space of functions $[0,1) \to \{0,1\}$ with the product topology (where $\{0,1\}$ is discrete). Note that every $x \in [0,1)$ has a unique binary expansion

$$x = \sum_{i \geq 1} \frac{a_i(x)}{2^i}$$

with each $a_i(x) \in \{0,1\}$ that does not end in an infinite string of 1's. For $n \in \mathbb{Z}^+$ and $x \in [0,1)$, define $f_n(x) = a_n(x)$ so that $f_n \in X$.

(a) Draw the graph of f_n for $n = 1,2,3,4$.
(b) Prove that no subsequence of $(f_n)_{n\geq 1}$ converges in X.
(c) Let $\mathscr{E}$ denote the eventuality filter of $(f_n)_{n\geq 1}$. By the ultrafilter lemma, $\mathscr{E}$ is contained in an ultrafilter $\mathscr{F}$. What two facts guarantee that $\mathscr{F}$ converges?
(d) (Optional) Can you say anything about the limit(s) of $\mathscr{F}$?
(e) (Optional) On p.67, your book claims that if X is compact, then every sequence has a convergent subsequence, and it claims that this is a consequence of the Bolzano–Weierstrass Theorem (p.48). What gives? [*Hint*: The claim on p.67 is false.]

Problem 3. (a) Suppose that

$$\cdots X_{-2} \subseteq X_{-1} \subseteq X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$$

is a chain of subspaces. After replacing each subset relation with its associated inclusion map, identify the limit and colimit of this diagram.

(b) For an integer k, let $k\colon \mathbb{Z} \to \mathbb{Z}$ denote the multiplication-by-k function. Prove that the colimit of the diagram

$$\mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{4} \mathbb{Z} \xrightarrow{5} \mathbb{Z} \xrightarrow{6} \cdots$$

in the category of Abelian groups is $\mathbb{Q}$, the group of rational numbers under addition. (Your explanation should include the explicit cone under this diagram into which $\mathbb{Q}$ fits.)

Problem 4. The real projective plane $\mathbb{R}P^2$ is the space of lines in $\mathbb{R}^3$. More formally,

$$\mathbb{R}P^2 = (\mathbb{R}^3 \smallsetminus \{0\})/{\sim}$$

where $(x,y,z) \sim (\lambda x, \lambda y, \lambda z)$ for all $\lambda \in \mathbb{R} \smallsetminus \{0\}$.

Let X denote the pushout

$$\begin{array}{ccc} S^1 & \xrightarrow{2} & S^1 \\ {\scriptstyle i}\downarrow & & \downarrow \\ D^2 & \longrightarrow & X \end{array}$$

where i is the inclusion of S^1 as the boundary of the disk D^2 and $2\colon S^1 \to S^1$ is the map $z \mapsto z^2$ (viewing S^1 as unit norm complex numbers). Prove that X is homeomorphic to $\mathbb{R}P^2$. [*Hint*: You might first express X as a particular quotient of D^2.]

[1] This was asserted but not proved in lecture.

Problem 5. Fix a prime number p and consider the inverse system

$$\cdots \mathbb{Z}/p^4\mathbb{Z} \to \mathbb{Z}/p^3\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$$

where the map $\mathbb{Z}/p^{r+1}\mathbb{Z} \to \mathbb{Z}/p^r\mathbb{Z}$ is the reduction homomorphism.[2] Let $\mathbb{Z}_p$ denote the subset of $\prod_{r\geq 1} \mathbb{Z}/p^r\mathbb{Z}$ consisting of sequences $(a_r)_{r\geq 1}$ such that $a_{r+1} \equiv a_r \pmod{p^r}$; these are the *p-adic integers*.

(a) Prove that $\mathbb{Z}_p$ is an Abelian group under coordinatewise addition.
(b) Prove that $\mathbb{Z}_p$ is the limit of the above inverse system in the category of Abelian groups. (Be explicit about the cone into which $\mathbb{Z}_p$ fits.)
(c) For $a \in \mathbb{N}$ and $1 \leq n \leq p^a$ an integer, define

$$U_a(n) = \{n + \lambda p^a \mid \lambda \in \mathbb{Z}_p\}$$

where p^a scales λ coordinatewise. Prove that the sets $U_a(n)$ form the basis of a topology on $\mathbb{Z}_p$. (Henceforth, we will always consider $\mathbb{Z}_p$ with this topology.)
(d) The *Cantor set* $\mathcal{C}$ may be described as the subspace of $[0, 1]$ consisting of real numbers that may be expressed in ternary with no 1's. Prove that $\mathbb{Z}_2 \cong \mathcal{C}$ as spaces. Use this to conclude that $\mathbb{Z}_2$ is compact.
(e) (Optional) Prove that $\mathbb{Z}_p$ has the topology of a Cantor set for all p.
(f) (Optional) Prove that $\mathbb{Z}_p$ is the limit of the $\mathbb{Z}/p^r\mathbb{Z}$ inverse system in Top (where each $\mathbb{Z}/p^r\mathbb{Z}$ has the discrete topology).[3]

[2] Apply the third isomorphism theorem to $p^{r+1}\mathbb{Z} \leq p^r\mathbb{Z} \leq \mathbb{Z}$ if you want to be formal about this.

[3] Limits of diagrams of finite discrete spaces are called *profinite spaces*, so this tells us that $\mathbb{Z}_p$ is a profinite space.