MATH 342: TOPOLOGY
HOMEWORK DUE FRIDAY WEEK 8

Problem 1. Suppose that X and Y are first countable spaces. Prove that a function $f\colon X \to Y$ is continuous if and only if for every sequence (x_n) in X with $(x_n) \to x$, the sequence $(f(x_n)) \to f(x)$.[1]

Problem 2. Suppose that X is a first countable space and that $A \subseteq X$. Prove that every $x \in \overline{A}$ is the limit of some sequence in A.[2]

Problem 3. A subset $A \subseteq X$ is called *dense* when its closure is all of X.

(a) Suppose that A is dense in X, Y is a Hausdorff space, and $f\colon A \to Y$ is a continuous function. Prove that there is at most one continuous function $g\colon X \to Y$ such that $g|_A = f$. (Such a g is called an *extension* of f to X.)

(b) Use (a) to prove that every *continuous* group homomorphism $(\mathbb{R}, +) \to (\mathbb{R}, +)$ is of the form $x \mapsto \lambda x$ for some $\lambda \in \mathbb{R}$.

Problem 4. A *directed set* is a pair $(S, \leq)$ with S a set and $\leq$ a relation on S which is reflexive and transitive[3] such that for all $s, t \in S$, there exists $u \in S$ such that $s \leq u$ and $t \leq u$.[4] A *net* in a space X is a function $\eta\colon S \to X$ whose domain is a directed set. We say that η converges to $x \in X$ if and only if its *eventuality filter*

$$\mathscr{E}_\eta = \{A \subseteq X \mid \text{there exists } t \in S \text{ such that } s \geq t \text{ implies } \eta(s) \in A\}$$

contains $\mathcal{T}_x$; in this case we write $\eta \to x$.

(a) Explain why every sequence is a net.[5]

(b) Given a proper filter $\mathscr{F}$, let

$$\mathscr{D} := \{(A, a) \in 2^X \times X \mid a \in A \in \mathscr{F}\}.$$

Show that $\mathscr{D}$ is directed by the relation $(A, a) \leq (B, b)$ if and only if $B \subseteq A$.

(c) Let $\pi_{\mathscr{F}}\colon \mathscr{D} \to X$ be the net defined by $\pi_{\mathscr{F}}(A, a) = a$. Prove that $\mathscr{E}_{\pi_{\mathscr{F}}} = \mathscr{F}$.

(d) Conclude that $\pi_{\mathscr{F}} \to x$ if and only if $\mathscr{F} \to x$. This means that nets and proper filters are formally interchangeable.

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[1] We proved — and you may cite — the forwards implication in class.

[2] In class, we saw that sequences in A converge to points of $\overline{A}$; this problem provides a converse statement.

[3] This makes $(S, \leq)$ as *preorder*.

[4] So a directed set is a preorder for which every pair of elements has an upper bound.

[5] Note that convergence of a sequence matches convergence of the associated net.