MATH 342: TOPOLOGY
HOMEWORK DUE FRIDAY WEEK 5

Problem 1. Reread Example 1.4 from the textbook and recall the *order topology* on a totally ordered set $(X, <)$. The *lexicographic ordering* $\prec$ on the unit square $[0,1]^2$ is defined by

$$(x, y) \prec (x', y') \iff x < x' \text{ or } x = x' \text{ and } y < y'.$$

(Here $<$ is the usual ordering on $\mathbb{R}$.) Call $[0,1]^2$ with the order topology induced by $\prec$ the *lexicographically ordereed square*.

(a) Check that $\prec$ is a total ordering on $[0,1]^2$ and then write "I have checked that $\prec$ is a total ordering on $[0,1]^2$."

(b) Endow each of the following subsets of the lexicographically ordereed square with the subspace topology. In each case, describe the topology on the set in more explicit/familiar terms (*e.g.*, discrete, *etc.*).
   - (i) $A = \{(x, 1/2) \mid x \in [0,1]\}$,
   - (ii) $B = \{(1/2, y) \mid y \in [0,1]\}$,
   - (iii) $C = \{(x, 1) \mid x \in [0,1]\}$,
   - (iv) $\Delta = \{(x, x) \mid x \in [0,1]\}$.

Problem 2. Given topological spaces X and Y, a function $f\colon X \to Y$ is called *open* (resp. *closed*) if and only if $fU \subseteq Y$ is open (resp. closed) for all open (resp. closed) $U \subseteq X$. Suppose that $f\colon X \to Y$ is a continuous surjection.

(a) Give an example to show f may be open but not closed.

(b) Give an example to show f may be closed but not open.

(c) Prove that if f is open or closed, then the topology on Y is the same as the quotient topology on Y induced by f. (For notational convenience, you may want to refer to these as $\mathcal{T}_Y$ and $\mathcal{T}_f$, respectively; your task is to show $\mathcal{T}_Y = \mathcal{T}_f$ for f open or closed.)

Problem 3. Let $\{X_\alpha \mid \alpha \in A\}$ be a collection of topological spaces and recall that the product topology on $X = \prod_{\alpha \in A} X_\alpha$ is defined to be the topology generated by

$$\mathscr{B} = \left\{ \prod_{\alpha \in A} U_\alpha \,\middle|\, U_\alpha \subseteq X_\alpha \text{ open and all but finitely many } U_\alpha = X_\alpha \right\}.$$

(a) Prove that $\mathscr{B}$ is a basis.

(b) Prove that the product topology on X is the coarsest topology on X for which all the projection maps $\pi_\alpha\colon X \to X_\alpha$ are continuous.

(c) Prove that the product topology on X satisfies and is characterized by the following universal property: a function $f\colon Z \to X$ is continuous if and only if for every $\alpha \in A$, the function $\pi_\alpha f\colon Z \to X_\alpha$ is continuous.

(d) Briefly explain why the universal property may be sloganized as "a function into a product is continuous if and only if each of its components is continuous."

Problem 4. Let ℓ_2 denote the set of *square-summable* sequences of real numbers, *i.e.*,

$$\ell_2 := \left\{ (a_n)_{n \in \mathbb{N}} \,\middle|\, a_n \in \mathbb{R},\ \sum a_n^2 < \infty \right\}.$$

Define a function $d\colon \ell_2 \times \ell_2 \to \mathbb{R}$ by the rule

$$d(a,b) = \sqrt{\sum (b_n - a_n)^2}.$$

In this problem, you may assume that d is a well-defined metric on ℓ_2. (See Examples 1.7 and 1.8 in the textbook for additional context.) Endow ℓ_2 with the metric topology induced by d.

Separately, we may give the set $\mathbb{R}^{\mathbb{N}}$ of real-valued sequences the product topology. Note that $\ell_2 \subseteq \mathbb{R}^{\mathbb{N}}$.

(a) Prove that the subspace topology on ℓ_2 induced by $\ell_2 \subseteq \mathbb{R}^{\mathbb{N}}$ is *not* the same as the topology induced by d.

(b) The *Hilbert cube* is the set

$$H := [0,1] \times [0,1/2] \times [0,1/3] \times [0,1/4] \times [0,1/5] \times \cdots .$$

Show that $H \subseteq \ell_2$.

(c) Give H the subspace topology inside of ℓ_2. Prove that

$$H \cong [0,1]^{\mathbb{N}}$$

where the latter space is a product of countably many unit intervals[1] with the product topology.

Problem 5. Given a topological space X and $n \in \mathbb{N}$, prove that any two functions $f, g\colon X \to \mathbb{R}^n$ are homotopic.

[1]You could also think of $[0,1]^{\mathbb{N}}$ as sequences $(a_n)_{n\in\mathbb{N}}$ with $0 \le a_n \le 1$ for all n.