MATH 342: TOPOLOGY
HOMEWORK DUE FRIDAY WEEK 3

Problem 1. A morphism $f\colon X \to Y$ in a category C is called *monic* (or a *monomorphism*) when it satisfies the following property:

$$\text{if } g_1, g_2\colon Z \to X \text{ are morhisms in } \mathsf{C} \text{ such that } fg_1 = fg_2, \text{ then } g_1 = g_2.$$

A morphism $f\colon X \to Y$ in a category C is called *epic* (or an *epimorphism*) when it satisfies the following property:

$$\text{if } g_1, g_2\colon Y \to Z \text{ are morphisms in } \mathsf{C} \text{ such that } g_1 f = g_2 f, \text{ then } g_1 = g_2.$$

(a) Which morphisms in Set are monic? which are epic?
(b) Prove that every left invertible morphism is monic, and that every right invertible morphism is epic.
(c) Find morphisms (in particular categories) exhibiting that the converses to the statements in (b) are *not* true in general.
(d) Give an example of a morphism in Top that is epic and monic but not an isomorphism.

Problem 2. (a) What is a functor $BG \to BH$ for G and H groups?
(b) Given a partially ordered set $(P, \leq)$, continue to write P for the associated category. If $(P, \leq)$ and $(Q, \leq)$ are partially ordered sets, what is a functor $P \to Q$?

Problem 3. We know that functors preserve isomorphisms. Find an example to demonstrate that functors need not *reflect isomorphisms*: that is, find a functor $F\colon \mathsf{C} \to \mathsf{D}$ and a morphism f in C such that Ff is an isomorphism in D but f is not an isomorphism in C.

Problem 4. Check that there are four distinct topologies on a two-element set and that there are 29 distinct topologies on a three-element set. In each case, draw pictures of each topology and arrange them into *Hasse diagrams*[1] for the associated partially ordered set of topologies under containment.[2]

Problem 5. Give an example of a path $p\colon [0,1] \to X$ connecting a to b in the topological space $X = (\{a,b,c,d\}, \mathcal{T})$ where

$$\mathcal{T} = \{\varnothing, \{a\}, \{c\}, \{a,c\}, \{a,b,c\}, \{a,c,d\}, \{a,b,c,d\}\}.$$

[1] A *Hasse diagram* for a poset $(P, \leq)$ is a directed graph with vertex set P and a directed edge $p \to q$ if and only if $p < q$ in P and there is no $r \in P$ such that $p < r < q$.

[2] *Challenge*: If the sets in question are $[2] = \{1,2\}$ and $[3] = \{1,2,3\}$, then the posets of topologies on $[2]$ and $[3]$ are sub-posets of $2^{2^{[2]}}$ and $2^{2^{[3]}}$, respectively. The Hasse diagrams for $2^{2^{[2]}}$ and $2^{2^{[3]}}$ are 4- and 8-dimensional cubes, respectively. Find a way to represent / think about the Hasse diagrams of topologies inside of these cubes.