MATH 342: TOPOLOGY
HOMEWORK DUE FRIDAY WEEK 2

Problem 1. Suppose $\mathcal{S} \subseteq 2^X$ with union all of X. Show that $\langle \mathcal{S} \rangle$ has basis the set of finite intersections of elements of $\mathcal{S}$. Conclude that $\langle \mathcal{S} \rangle$ consists of arbitrary unions of finite intersections of elements of $\mathcal{S}$. (Such a set $\mathcal{S}$ is called a *subbasis*.)

Problem 2. Prove that d_1 and d_∞ are metrics on $\mathbb{R}^n$. Draw the unit ball centered at $0 \in \mathbb{R}^2$ for each of these metrics.

Problem 3. Suppose that d_1 and d_2 are metrics on X with associated topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively.

(a) Show that $\mathcal{T}_1 = \mathcal{T}_2$ if and only if for all $x \in X$ and $r > 0$, there exists $r', r'' > 0$ such that

$$B_{d_1}(x, r') \subseteq B_{d_2}(x, r) \quad \text{and} \quad B_{d_2}(x, r'') \subseteq B_{d_1}(x, r).$$

(Here $B_d(x, r) = \{y \in X \mid d(x, y) < r\}$ is the open ball centered at x of radius r with respect to metric d.)

(b) Show that d_1, d_2, and d_∞ define the same topology on $\mathbb{R}^2$. (Or go further and show that every metric topology on $\mathbb{R}^n$ given by d_p, $1 \leq p \leq \infty$, is the same.)

Problem 4. Suppose that $f\colon X \to Y$ is a function between metric spaces X and Y and that for all $x \in X$, $y \in Y$ such that $f(x) = y$, the following condition holds:

$$\text{for all } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that} fB(x, \delta) \subseteq B(y, \varepsilon).$$

(a) Briefly explain the similarity between this statement and the definition of continuity you see in Math 112.

(b) Prove that f is continuous (in the sense that $f^{-1}U$ is open in X for all $U \subseteq Y$ open, where X and Y carry their respective metric topologies).

Problem 5. Consider a morphism $f\colon x \to y$ in a category C. Show that if there exist morphisms $g, h\colon y \rightrightarrows x$ such that $gf = \mathrm{id}_x$ and $fh = \mathrm{id}_y$, then $g = h$ and f is an isomorphism. Use this to show that inverse morphisms are unique.

Problem 6. Fix a field k and let Mat_k have objects the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ and morphisms

$$\mathsf{Mat}_\mathsf{k}(m, n) = \{n \times m \text{ matrices with entries in } \mathsf{k}\}.$$

Define composition by

$$\begin{aligned} \mathsf{Mat}_\mathsf{k}(n, \ell) \times \mathsf{Mat}_\mathsf{k}(m, n) &\longrightarrow \mathsf{Mat}_\mathsf{k}(m, \ell) \\ (A, B) &\longmapsto AB \end{aligned}$$

where AB denotes the matrix product of A and B. Check that Mat_k is a category, freely citing results needed from linear algebra.