MATH 341: TOPICS IN GEOMETRY
HOMEWORK DUE MONDAY WEEK 9

Problem 1. Verify that the definition of the cross ratio given in the course notes is well-defined and independent of choices. (Part of your work in this problem is to interpret precisely what that means!)

Problem 2. Let P, Q, R be three distinct points of $\hat{\mathbb{C}}$. Show that the point $S \in \hat{\mathbb{C}}$ belongs to the circle through P, Q, R if and only if $[P, Q; R, S] \in \hat{\mathbb{R}}$.

Problem 3. For $\sigma \in \mathrm{PGL}_2(\mathbb{C})$, let $\mathrm{Fix}(\sigma)$ denote the set of fixed points for the action of σ on $\hat{\mathbb{C}}$. Suppose that $\sigma, \tau \in \mathrm{PGL}_2(\mathbb{C}) \setminus \{\mathrm{id}\}$ and that $\mathrm{Fix}(\sigma) = \mathrm{Fix}(\tau)$. Prove that $\sigma\tau = \tau\sigma$. *Bonus*: Determine necessary and sufficient conditions on fixed points such that $\sigma\tau = \tau\sigma$.

Problem 4. Show that the evaluation map

$$\begin{aligned} \mathrm{PGL}_2(\mathbb{C}) &\longrightarrow \hat{\mathbb{C}} \times \hat{\mathbb{C}} \times \hat{\mathbb{C}} \\ \sigma &\longmapsto (\sigma(\infty), \sigma(0), \sigma(1)) \end{aligned}$$

is a bijection of $\mathrm{PGL}_2(\mathbb{C})$ onto the open subset $\mathrm{Conf}_3(\hat{\mathbb{C}})$ of $\hat{\mathbb{C}}^3$ consisting of triples of distinct points of $\hat{\mathbb{C}}$. (The space $\mathrm{Conf}_3(\hat{\mathbb{C}})$ is called the *ordered configuration space* of 3 points in $\hat{\mathbb{C}}$.) *Bonus*: Show that the map is a homeomorphism.

Problem 5. Show that $\sigma \in \mathrm{GL}_2(\mathbb{R})$ acts on the Poincaré half-plane H^2 as reflection in a geodesic if and only if $\det \sigma < 0$ and $\mathrm{tr}\, \sigma = 0$.