MATH 341: TOPICS IN GEOMETRY
HOMEWORK DUE FRIDAY WEEK 5

SOLUTION HINTS

Problem 1. Let $\mathbb{H}^1$ denote the upper sheet of $S(\mathbb{R}^{-1,1})$. Set $c = (\sqrt{3}, \sqrt{2}) \in \mathbb{R}^{-1,1}$ and note that it has norm -1. Explicitly compute τ_c, reflection along c in $\mathbb{R}^{-1,1}$, and draw a picture exhibiting how it acts on $\mathbb{H}^1$.

Solution. We have $\langle c, c\rangle = 2 - 3 = -1$, as claimed. The formula for τ_c is

$$\tau_c(p) = p - 2\frac{\langle p, c\rangle}{\langle c, c\rangle}c = p + 2\langle p, c\rangle c.$$

If $p = (x, t)$, then we can rewrite this as

$$\tau_c(x, t) = (x, t) + 2(\sqrt{2}t - \sqrt{3}x)(\sqrt{3}, \sqrt{2}) = (-5x + 2\sqrt{6}t, -2\sqrt{6}x + 5t).$$

One can compute the value of τ_c on a few points of $\mathbb{H}^1$ to see that it "folds" $\mathbb{H}^1$ over the fixed point $(\sqrt{2}, \sqrt{3})$ (which is the unique point in the intersection of $\mathbb{H}^1$ with $c^\perp$). This exercise hopefully disabuses the reader from thinking too naïvely about the nature of reflections in Minkowski space! □

Problem 2. Let D^2 denote the 2-dimensional Klein disk. Draw a family of circles along a diameter of D^2 where the radius of each circle is 1 (measured via the Klein hyperbolic metric). Justify your calculations and picture. Conclude by observing that while lines are easy to visualize in D^2, circles are less pleasant.

Solution. You should get a family of ellipses. This illustrates that the Klein disk is not a conformal model of hyperbolic space. □

Problem 3. Verify the formulæ for $p\colon D^n \to \mathbb{H}^n$ and $f\colon D^n \to \mathbb{H}^n$ given in the notes. (These maps are defined in terms of intersections between certain lines and a particular model of $\mathbb{H}^n$. You need to check that the intersection points are in fact given by the formulæ in the notes.)

Solution. In both cases, it is easy to check that the values lie on the appropriate ray. Calculating norms implies that $p(x)$ and $f(x)$ lie on $\mathbb{H}^n$. □

Problem 4. Prove Ptolemy's theorem:

> Let E be a Euclidean vector space of dimension n. Then $n+2$ points $x_1, \ldots, x_{n+2} \in E$ lie on a sphere if and only if $\det(d(x_i, x_j)^2)_{i,j} = 0$, where d denotes Euclidean distance.

Hint: Show that $\langle \iota x, \iota y\rangle = \frac{1}{2}d(x, y)^2$ for $x, y \in E$.

Solution. The hint is an easy calculation, and it implies that the determinant in question is 0 if and only if $\det(\langle \iota x_i, \iota x_j\rangle) = 0$. Since $\langle -, -\rangle$ is a regular quadratic form on $E \oplus \mathbb{R}^2$, this "Gram-type" matrix has determinant 0 if and only if $\operatorname{span}\{\iota x_1, \ldots, \iota x_{n+2}\}$ has dimension less than $n+2$. But this is equivalent to the existence of $c \in E \oplus \mathbb{R}^2$ such that $\langle \iota x_i, c\rangle = 0$ for all x_i, and we may take c to have norm -1 (check!). This condition is equivalent to $\tau_c \in \operatorname{Lor}(E \oplus \mathbb{R}^2)$ fixing ιx_i for all i (check by looking at the formula for τ_c), which is in turn equivalent to the inversion $\sigma = \iota\tau_c\iota^{-1} \in \text{Möb}(E)$

fixing x_i for all x_i. A quantitative version of Corollary 7.12 of Iversen implies that this is equivalent to σ being inversion in a sphere, and this sphere contains $x_1, \ldots, x_{n+2}$. $\square$

As an amusing diversion, consider the case in which E has dimension 2 and the pairwise distances between points x_1, x_2, x_3, x_4 are a, b, c, d, e, f. The matrix from Problem 4 is then

$$\begin{pmatrix} 0 & a^2 & e^2 & d^2 \\ a^2 & 0 & b^2 & f^2 \\ e^2 & b^2 & 0 & c^2 \\ d^2 & f^2 & c^2 & 0 \end{pmatrix}$$

with determinant

$$\begin{aligned} & a^4c^4 - 2a^2b^2c^2d^2 + b^4d^4 - 2a^2c^2e^2f^2 - 2b^2d^2e^2f^2 + e^4f^4 \\ & \qquad = (ac - bd - ef)(ac + bd - ef)(ac - bd + ef)(ac + bd + ef). \end{aligned}$$

We conclude that x_1, x_2, x_3, x_4 lie on a circle if and only if

$$ac = bd + ef \quad \text{or} \quad ef = ac + bd \quad \text{or} \quad bd = ac + ef \quad \text{or} \quad ac + bd + ef = 0,$$

a version of the classical Ptolemy theorem.