MATH 341: TOPICS IN GEOMETRY
HOMEWORK DUE FRIDAY WEEK 4

SOLUTION HINTS

Problem 1. This is a continuation of Problem 5 from HW2, and you may freely use the conclusions of that problem here.

Let $\triangle ABC$ be a triangle in the Euclidean plane. Let A^* denote the point in the plane such that $\triangle A^*BC$ is equilateral and such that A and A^* lie on opposite sides of the line through B and C. Let θ_A denote the rotation about the circumcenter[1] O_A of $\triangle A^*BC$ which takes B to C. Similarly define θ_C and θ_B.

(a) Show that $\theta_C\theta_B\theta_A = \mathrm{id}$. *Hint*: Show that $\theta_C\theta_B\theta_A$ is a rotation by angle 0.
(b) Extend the notation above in the natural way and then show that the circumcenters O_A, O_B, and O_C form an equilateral triangle. *Hint*: Calculate $\theta_B\theta_A$.

Solution. (a) The rotations θ_B and θ_A are both by an angle of $2\pi/3$, so $\theta_B\theta_A$ is a rotation by $4\pi/3$ or $-2\pi/3$. Since θ_C is a rotation by $2\pi/3$, the composite $\theta_C\theta_B\theta_A$ is a rotation by 0 and hence the identity.
(b) Use Problem 5(c) from the previous homework assignment.

□

Problem 2. Let E denote an n-dimensional Euclidean space and let k and ℓ be lines through the origin in E. Define the *acute angle* $\angle(k,\ell) \in [0,\pi/2]$ between k and ℓ by the formula

$$\angle(k,\ell) = \arccos\left|\left\langle \hat{k},\hat{\ell}\right\rangle\right|$$

where $\hat{k}$ and $\hat{\ell}$ are unit vectors generating k and ℓ, respectively.

(a) Show that the acute angle defines a metric on *projective space* $P(E)$, the set of lines through the origin in E. *Hint*: Interpret $P(E)$ as the orbit space for the action of the antipodal map on the sphere $S(E)$, and use what we have developed about spherical geometry.
(b) Show that there is a unique geodesic passing through any two distinct points of $P(E)$.

Solution. (a) The map $S(E) \to P(E)$, $x \mapsto \mathrm{span}\{x\}$ is surjective and 2-to-1 with fiber $\{\hat{\ell},-\hat{\ell}\}$ over a line $\ell \in P(E)$. This identifies $P(E)$ with $S(E)/x \sim -x$. Identity of indiscernibles and symmetry of $\angle(k,\ell)$ are easy to check. For the triangle inequality, one must *carefully* choose lifts to $S(E)$ (so that acute angles are chosen) then apply the spherical triangle inequality. **Many students failed to use sufficient care in selecting lifts.**
(b) Again by carefully choosing $\hat{k}$ and $\hat{\ell}$ with $\left\langle \hat{k},\hat{\ell}\right\rangle \geq 0$, one can project the spherical geodesic from $\hat{k}$ to $\hat{\ell}$ down to $P(E)$ to get a geodesic from k to ℓ in $P(E)$. For uniqueness, lift geodesics to $S(E)$ to contradict spherical geodesic uniqueness.

□

Problem 3. The *torus* is the set $S^1 \times S^1$ endowed with the product metric from Problem 4 of the previous problem set (where $S^1 = S(\mathbb{R}^2)$ is endowed with the standard spherical metric). The

[1]The *circumcenter* of a triangle is the center of the (unique) circle passing through the vertices of the triangle.

canonical map $\chi\colon \mathbb{R} \to S^1$, $\theta \mapsto (\cos\theta, \sin\theta)$ induces a map

$$\begin{aligned} \nu\colon \mathbb{R}^2 &\longrightarrow S^1 \times S^1 \\ (\theta, \varphi) &\longmapsto (\chi(\theta), \chi(\varphi)). \end{aligned}$$

(a) For points $p, q \in S^1 \times S^1$, show that

$$d(p,q) = \inf_{\{(P,Q)\in\mathbb{R}^2\times\mathbb{R}^2 \mid p=\nu(P), q=\nu(Q)\}} d(P,Q).$$

(b) Show that ν is a *local isometry*: there exists a constant $r > 0$ such that for all $P \in \mathbb{R}^2$ the open disk with center P and radius r is mapped bijectively by ν onto the "metric disk" $\{q \in S^1 \times S^1 \mid d(q, \nu(P)) < r\}$ in $S^1 \times S^1$.

(c) Show that an affine line in $\mathbb{R}^2$ is mapped by ν onto a geodesic in $S^1 \times S^1$ and that all geodesics in $S^1 \times S^1$ have this form.

Solution. (a) The crux here is to choose $\theta_0, \theta_1, \phi_0, \phi_1 \in \mathbb{R}$ such that $p = \nu(\theta_0, \phi_0)$, $q = \nu(\theta_1, \phi_1)$, $|\theta_0 - \theta_1| < \pi$, and $|\phi_0 - \phi_1| < \pi$. Taking $P = (\theta_0, \phi_0)$ and $Q = (\theta_1, \phi_1)$ gives $d(p,q) = d(P,Q)$, so the infimum in question is $\geq d(p,q)$. Other valid choices of P and Q add integer multiples of 2π to the angles and a bit of formula/inequality manipulation gives the other needed inequality.

(b) The above argument makes it clear that $r \leq \pi/2$ will work.

(c) This mostly amounts to using (b) and the fact that affine lines in $\mathbb{R}^2$ are geodesics.

$\square$