MATH 341: TOPICS IN GEOMETRY
HOMEWORK DUE FRIDAY WEEK 3

SOLUTION HINTS

Warning: These are not full solutions. Your work should be more complete and more rigorous.

Problem 1. Let E denote three-dimensional Euclidean space and let B denote the closed ball of radius π centered at the origin in E. Define a function $f\colon B \to \mathrm{SO}(E)$ taking a point x to the rotation with axis $\mathrm{span}_{\mathbb{R}}\{x\}$ by angle $|x|$ (in the counterclockwise direction relative to the orientation of $\mathrm{span}_{\mathbb{R}}\{x\}$ pointing from 0 to x).

(a) Show that f is injective on $B^\circ = \{x \in E \mid |x| < \pi\}$.
(b) Suppose that $\sigma \in \mathrm{SO}(E)$ is the value of f on a point $x \in \partial B = \{x \in E \mid |x| = \pi\}$. Precisely describe $f^{-1}\{\sigma\}$.
(c) (Optional.) Real projective 3-space, $\mathbb{R}P^3$, is the set of lines through the origin in $\mathbb{R}^4$. Explain (perhaps informally) why your answers to (a) and (b) imply that $\mathrm{SO}(3)$ is homeomorphic to $\mathbb{R}P^3$.

Solution. (a) Suppose that $f(x) = f(y)$ for $x, y \in B^\circ$. Then $f(x)$ and $f(y)$ have the same axis of rotation, whence $\mathrm{span}\{x\} = \mathrm{span}\{y\}$. Since $f(x)$ and $f(y)$ rotate E by the same angle, we also have $|x| = |y| + 2\pi k$ for some $k \in \mathbb{Z}$. Since $x, y \in B^\circ$, we deduce that $k = 0$, so $x = \pm y$. It is easy to check that $f(x) \neq f(-x)$ for $x \in B^\circ$, so $x = y$, proving injectivity on B°.
(b) The previous argument reveals that $f^{-1}\{\sigma\} = \{\pm x\}$.
(c) Let $\sim$ be the equivalence relation on B generated by $x \sim -x$ for $x \in \partial B$. We need to show that $B/\sim \cong \mathbb{R}P^3$. We take it as clear that $\mathbb{R}P^3$ is identified with S^3 mod the antipodal action. We can map B to the "upper hemisphere" of S^3, and every point is in its own equivalence class except for those on the boundary.

□

Problem 2. Let E denote three-dimensional Euclidean space and let $\sigma \in \mathrm{SO}(E)$ be a rotation with angle $\geq \pi/2$. Show that there exists a line L through 0 such that L and $\sigma(L)$ are orthogonal. *Hint*: Observe that the angle between L and $\sigma(L)$ is a continuous function of L.

Solution. If L is the axis of σ, then $\angle(L, \sigma(L)) = 0$. If L is perpendicular to the axis of σ, then $\angle(L, \sigma(L)) \geq \pi/2$. If we can argue that $\angle(L, \sigma(L))$ is a continuous function of L, and we can "draw a line" between the above two lines, then we may apply the intermediate value theorem to conclude that an L exists such that $\angle(L, \sigma(L)) = \pi/2$. For full credit, continuity and the "line between lines" must be formalized. □

Problem 3. Let $E = \mathbb{R}^{-3,1}$ denote the standard four-dimensional real vector space with quadratic form of Sylvester type $(-3, 1)$. (In particular, $(x, y, z, t) \mapsto t^2 - x^2 - y^2 - z^2$.) Show that the diagonal matrices $P = \mathrm{diag}(-1, -1, -1, 1)$ and $T = -P = \mathrm{diag}(1, 1, 1, -1)$ are in $\mathrm{O}(-3, 1) := O(\mathbb{R}^{-3,1})$. Determine the structure of the subgroup of $\mathrm{O}(-3, 1)$ generated by P and T. Which elements of this group belong to $\mathrm{Lor}^+(E)$?

Solution. We have $P^2 = T^2 = I$ and $PT = -I$. We conclude that the subgroup generated by P and T is $\{I, P, T, -I\}$ and is isomorphic to the Klein four-group. Only I belongs to $\mathrm{Lor}^+(E)$, which can be checked on determinants and the $\langle x, \sigma(x)\rangle > 0$ condition. □

Problem 4. Let X and Y be metric spaces with distance functions d_X and d_Y, respectively. For points $P = (x, y)$ and $Q = (x', y')$ in $X \times Y$, define

$$d(P, Q) = \sqrt{d_X(x, x')^2 + d_Y(y, y')^2}.$$

Prove that the associated function $d\colon (X \times Y) \times (X \times Y) \to \mathbb{R}$ gives a metric on $X \times Y$.

Solution. If you Google "product metric space" you will find many solutions to this problem. *Exercise*: Determine which solutions are actually correct. □

Problem 5. Let σ and ρ be rotations of the Euclidean plane with distinct centers A and B.

(a) Let γ denote reflection in the line through A and B. Show that there is a line a through A and a line b through B such that $\rho = \beta\gamma$ and $\sigma = \gamma\alpha$ where β is reflection in b and α is reflection in a.

(b) In the notation of (a), show that if a and b are parallel, then $\rho\sigma = \beta\alpha$ is a translation; if a and b intersect in C, then $\rho\sigma = \beta\alpha$ is a rotation with center C.

(c) Suppose that ρ and σ are rotations with the same angle $-2\pi/3$ (still with distinct centers A and B). Show that $\rho\sigma$ is a rotation with angle $2\pi/3$ and that its center C forms an equilateral triangle with A and B.

Please draw pictures throughout your answer to Problem 5.

Solution. Begin by computing the rotation given by composition of reflection in lines through a common point. (You'll get a particular rotation involing twice the angle between the lines.) The rest follows with care and good draftsmanship. □