# MATH 341: TOPICS IN GEOMETRY
# HOMEWORK DUE FRIDAY WEEK 3

*Problem* 1. Let $E$ denote three-dimensional Euclidean space and let $B$ denote the closed ball of radius $\pi$ centered at the origin in $E$. Define a function $f\colon B \to \mathrm{SO}(E)$ taking a point $x$ to the rotation with axis $\mathrm{span}_{\mathbb{R}}\{x\}$ by angle $|x|$ (in the counterclockwise direction relative to the orientation of $\mathrm{span}_{\mathbb{R}}\{x\}$ pointing from $0$ to $x$).

(a) Show that $f$ is injective on $B^\circ = \{x \in E \mid |x| < \pi\}$.

(b) Suppose that $\sigma \in \mathrm{SO}(E)$ is the value of $f$ on a point $x \in \partial B = \{x \in E \mid |x| = \pi\}$. Precisely describe $f^{-1}\{\sigma\}$.

(c) (Optional.) Real projective 3-space, $\mathbb{R}P^3$, is the set of lines through the origin in $\mathbb{R}^4$. Explain (perhaps informally) why your answers to (a) and (b) imply that $\mathrm{SO}(3)$ is homeomorphic to $\mathbb{R}P^3$.

*Problem* 2. Let $E$ denote three-dimensional Euclidean space and let $\sigma \in \mathrm{SO}(E)$ be a rotation with angle $\geq \pi/2$. Show that there exists a line $L$ through $0$ such that $L$ and $\sigma(L)$ are orthogonal. *Hint*: Observe that the angle between $L$ and $\sigma(L)$ is a continuous function of $L$.

*Problem* 3. Let $E = \mathbb{R}^{-3,1}$ denote the standard four-dimensional real vector space with quadratic form of Sylvester type $(-3, 1)$. (In particular, $(x, y, z, t) \mapsto t^2 - x^2 - y^2 - z^2$.) Show that the diagonal matrices $P = \mathrm{diag}(-1, -1, -1, 1)$ and $T = -P = \mathrm{diag}(1, 1, 1, -1)$ are in $\mathrm{O}(-3, 1) := O(\mathbb{R}^{-3,1})$. Determine the structure of the subgroup of $\mathrm{O}(-3, 1)$ generated by $P$ and $T$. Which elements of this group belong to $\mathrm{Lor}^+(E)$?

*Problem* 4. Let $X$ and $Y$ be metric spaces with distance functions $d_X$ and $d_Y$, respectively. For points $P = (x, y)$ and $Q = (x', y')$ in $X \times Y$, define

$$d(P, Q) = \sqrt{d_X(x, x')^2 + d_Y(y, y')^2}.$$

Prove that the associated function $d\colon (X \times Y) \times (X \times Y) \to \mathbb{R}$ gives a metric on $X \times Y$.

*Problem* 5. Let $\sigma$ and $\rho$ be rotations of the Euclidean plane with distinct centers $A$ and $B$.

(a) Let $\gamma$ denote reflection in the line through $A$ and $B$. Show that there is a line $a$ through $A$ and a line $b$ through $B$ such that $\rho = \beta\gamma$ and $\sigma = \gamma\alpha$ where $\beta$ is reflection in $b$ and $\alpha$ is reflection in $a$.

(b) In the notation of (a), show that if $a$ and $b$ are parallel, then $\rho\sigma = \beta\alpha$ is a translation; if $a$ and $b$ intersect in $C$, then $\rho\sigma = \beta\alpha$ is a rotation with center $C$.

(c) Suppose that $\rho$ and $\sigma$ are rotations with the same angle $-2\pi/3$ (still with distinct centers $A$ and $B$). Show that $\rho\sigma$ is a rotation with angle $2\pi/3$ and that its center $C$ forms an equilateral triangle with $A$ and $B$.

Please draw pictures throughout your answer to Problem 5.