MATH 341: TOPICS IN GEOMETRY
HOMEWORK DUE FRIDAY WEEK 2

SOLUTION HINTS

Warning: These are not full solutions. Your work should be more complete and more rigorous.

Problem 1. For $r \in \mathbb{Z}^+$, determine formulæ for the combinatorial area and circumference of the polygonal disk of radius r that has 6 equilateral triangles around each vertex.

Answer. By basic counting or geometric arguments, one gets $A(r) = 6r^2$ and $C(r) = 6r$. Note that $A(r)/C(r) \to \infty$ as $r \to \infty$. □

Problem 2. For $r \in \mathbb{Z}^+$, determine formulæ and/or interesting bounds for the combinatorial area and circumference of the polygonal disk of radius r that has 7 equilateral triangles around each vertex.

Hint. By carefully drawing one-seventh of the polygonal disk of radius r, one can find the recurrence $C(r+1) = 3C(r) - C(r-1)$ with initial conditions $C(1) = 7$, $C(2) = 21$. This leads to $C(r) = 7F_{2r}$, where F_n is the n-th Fibonacci number. The combinatorial area is also an exponential function of r. In particular, $A(r)/C(r) \to \sqrt{5}$ as $r \to \infty$. □

Problem 3. Prove that geodesics in the Poincaré disk model satisfy the incidence axiom. Draw a picture of geodesics in the Poincaré disk exhibiting that this model does not satisfy the parallel axiom.

Proof. For the first part, you can explicitly construct the cirle through $P, Q \in D$ perpendicular to ∂D (and exhibit that no others exist), or proceed via an intermediate value theorem argument to prove existence.

For the second part, it suffices to make a simple drawing consisting of two geodesics intersecting in a point where both geodesics are on one side of a diameter of D. □

Problem 4. For the purposes of this problem, define a *hyperbolic plane* over a field k (with $\operatorname{char} \mathsf{k} \neq 2$) to be a quadratic form (V, P) which has a basis consisting of two isotropic vectors u, v with $\langle u, v \rangle \neq 0$.

(a) Show that all hyperbolic planes over k are isometric.
(b) Let (E, Q) be a nonsingular quadratic form which contains an isotropic vector $v \neq 0$. Show that v is contained in a hyperbolic plane $V \subseteq E$. (*Hint*: Choose $w \in E$ with $\langle v, w \rangle = 1$ and observe that $u = 2w - \langle w, w \rangle\, v$ is isotropic with $\langle v, u \rangle = 2$.)
(c) Under the assumptions of (b), show that the equation $Q(x) = a$ has a solution $x \in E$ for all $a \in \mathsf{k}$. (A form which attains all values in k is called *universal*, so this proves that any nonsingular quadratic form containing a nonzero isotropic vector is universal.)

Hint. For part (a), note that the Gram matrix of any hyperbolic plane takes the form $\begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}$ with respect to the basis u, v. The computation

$$\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}^{\top} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}$$

implies that all hyperbolic planes are isometric to the one with $\lambda = 1$. □

Problem 5. Show that $\det\colon \mathrm{Mat}_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ is a quadratic form (where $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ is the 4-dimensional $\mathbb{R}$-vector space of 2×2 real matrices), and that its polarization is given by

$$\langle A, B \rangle = \frac{1}{2} \operatorname{Tr}(AB^\vee)$$

where $B^\vee$ denotes the cofactor matrix of B given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\vee = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Show that the Sylvester type of $\det$ on $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ is $(-2, 2)$. Finally, show that

$$\langle A, B \rangle = \frac{1}{2}(\operatorname{Tr} A \operatorname{Tr} B - \operatorname{Tr} AB).$$

Hint. Most of this problem consists of simple comutations and verifications with 2×2 matrices. One can utilize the orthonormal basis $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ to check that the Sylvester type is $(-2, 2)$. (Note that the standard basis $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ldots$ is *not* orthogonal!) $\square$