MATH 341: TOPICS IN GEOMETRY
HOMEWORK DUE FRIDAY WEEK 2

Problem 1. For $r \in \mathbb{Z}^+$, determine formulæ for the combinatorial area and circumference of the polygonal disk of radius r that has 6 equilateral triangles around each vertex.

Problem 2. For $r \in \mathbb{Z}^+$, determine formulæ and/or interesting bounds for the combinatorial area and circumference of the polygonal disk of radius r that has 7 equilateral triangles around each vertex.

Problem 3. Prove that geodesics in the Poincaré disk model satisfy the incidence axiom. Draw a picture of geodesics in the Poincaré disk exhibiting that this model does not satisfy the parallel axiom.

Problem 4. For the purposes of this problem, define a *hyperbolic plane* over a field k (with $\operatorname{char} \mathsf{k} \neq 2$) to be a quadratic form (V, P) which has a basis consisting of two isotropic vectors u, v with $\langle u, v \rangle \neq 0$.

(a) Show that all hyperbolic planes over k is isometric.
(b) Let (E, Q) be a nonsingular quadratic form which contains an isotropic vector $v \neq 0$. Show that v is contained in a hyperbolic plane $V \subseteq E$. (*Hint*: Choose $w \in E$ with $\langle v, w \rangle = 1$ and observe that $u = 2w - \langle w, w \rangle\, v$ is isotropic with $\langle v, u \rangle = 2$.)
(c) Under the assumptions of (b), show that the equation $Q(x) = a$ has a solution $x \in E$ for all $a \in \mathsf{k}$. (A form which attains all values in k is called *universal*, so this proves that any nonsingular quadratic form containing a nonzero isotropic vector is universal.)

Problem 5. Show that $\det\colon \operatorname{Mat}_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ is a quadratic form (where $\operatorname{Mat}_{2\times 2}(\mathbb{R})$ is the 4-dimensional $\mathbb{R}$-vector space of 2×2 real matrices), and that its polarization is given by

$$\langle A, B \rangle = \frac{1}{2} \operatorname{Tr}(AB^\vee)$$

where $B^\vee$ denotes the cofactor matrix of B given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\vee = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Show that the Sylvester type of $\det$ on $\operatorname{Mat}_{2\times 2}(\mathbb{R})$ is $(-2, 2)$. Finally, show that

$$\langle A, B \rangle = \frac{1}{2}(\operatorname{Tr} A \operatorname{Tr} B - \operatorname{Tr} AB).$$