MATH 341: TOPICS IN GEOMETRY
EXAM 1

SOLUTION HINTS

Problem 1. Suppose that V is a 2-dimensional real vector space equipped with a symmetric bilinear form with Gram matrix

$$\begin{pmatrix} a & a \\ a & 1 \end{pmatrix}.$$

(a) For which $a \in \mathbb{R}$ is V nonsingular?
(b) For which $a \in \mathbb{R}$ does V have Sylvester type $(-1, 1)$?

Solution. (a) The form is nonsingular if and only if its determinant $a - a^2 = a(1 - a)$ is nonzero. This is the case only for $a \in \mathbb{R} \smallsetminus \{0, 1\}$.

(b) This is the case if and only if the form is nonsingular and its determinant is negative. In other words, $a \in \mathbb{R} \smallsetminus [0, 1]$.

□

Problem 2. Let S^2 be the unit sphere in standard Euclidean space $\mathbb{R}^3$. Define an *n-gon* on S^2 to be a sequence of n points along with the geodesic (shortest distance) arcs joining them (first point joined to second, second to third, ..., $(n-1)$-th to n-th, and n-th to first). Call the n-gon *simple* if the geodesic arcs do not intersect each other.

Let P be a simple n-gon on S^2 and let R be one of the two connected components of $S^2 \smallsetminus P$; call R the *interior* of P and let $\alpha_1, \dots, \alpha_n$ be the interior angles of P (*i.e.*, the angles between successive arcs measured inside of R). Assume that R is *spherically star convex*, meaning that there is a point $x \in R$ such that for all $y \in R$ there is a geodesic arc joining x to y which is contained in R. Under this hypothesis, determine the area of R in terms of $\alpha_1, \dots, \alpha_n$. (*Bonus*: Can you say something interesting about the area of R without the star convexity hypothesis?)

Solution. Use the star point to triangulate the polygon (drawing geodesic arcs from the star point to each vertex). Adding the areas of these triangles together (computed via Girard's theorem), we get

$$\sum_{i=1}^{n} \alpha_i - (n-2)\pi$$

as the area of the polygon. This is actually true for all polygons, but you have to be more careful with your triangulation. □

Problem 3. Let $D = D^2$ denote the Poincaré disk in $\mathbb{R}^2$ and let J denote the hemisphere $J = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1,\ z > 0\}$. Embed D into $\mathbb{R}^3$ as the set $D \times \{0\}$, and then let $f \colon D \to J$ denote stereographic projection from the point $(0, 0, -1)$. (This is the function taking a point $p \in D$ to the point $q \in J$ that the line joining $(0, 0, -1)$ to p hits.)

(a) Draw a picture of what f does to a generic point $p \in D$.
(b) Transport the Poincaré disk metric on D along f to produce an explicit metric on J. (Your answer should be a formula for a function $d \colon J \times J \to \mathbb{R}$ representing the transported metric. Note that (J, d) is now a new "hemispherical" model of the hyperbolic plane.)
(c) Give a concise, geometric description of the image of any geodesic $\gamma \colon \mathbb{R} \to J$.

Solution. (a) Pretty much an ice cream cone.
(b) You need to compute $f^{-1}\colon J \to D$ and then write something reasonable for

$$d_J(P,Q) = d_D(f^{-1}(P), f^{-1}(Q))$$

where $P, Q \in J$ and d_D is the hyperbolic metric on the Poincaré disk.
(c) These are circles in J lying in affine planes with normal vector having no z-component (*i.e.*, "vertical" planes). To see this, compute what f does to a circular arc in D perpendicular to ∂D.

□

Problem 4. Let E be a Euclidean space and define a *ball* in E to be a set of the form $\{x \in E \mid \langle x-c, x-c\rangle < r^2\}$ for some fixed $c \in E$ and $r \in \mathbb{R}_{>0}$. Give $E \oplus \mathbb{R}^2$ the symmetric bilinear form

$$\langle (e,a,b), (f,c,d)\rangle = -\langle e, f\rangle + \frac{1}{2}(ad+bc).$$

Let $N(E \oplus \mathbb{R}^2)$ be the vectors in $E \oplus \mathbb{R}^2$ with norm -1, and for $U \in N(E \oplus \mathbb{R}^2)$ define

$$\mathscr{D}_U = \{x \in E \mid \langle \iota x, U\rangle > 0\}$$

where $\iota\colon E \to E \oplus \mathbb{R}^2$ is given by $\iota(e) = (e, \langle e, e\rangle, 1)$.
(a) Show that every ball in E is of the form $\mathscr{D}_U$ for some $U \in N(E \oplus \mathbb{R}^2)$. (You may not assume the claims / definitions made between Corollaries 11.3 and 11.4 in the course notes and pp.35–36 of Iversen.)
(b) For which $U \in N(E \oplus \mathbb{R}^2)$ is the set $\mathscr{D}_U$ a ball?
(c) What is $\mathscr{D}_U$ when it is not a ball?

Solution. (a) A computation (perhaps inspired by the indicated portion of the notes) shows that $U = \frac{1}{r}(-c, r^2 - \langle c, c\rangle, -1)$ produces the ball of radius r centered at c.
(b) Again by computation, precisely those vectors $(e,a,b) \in N(E \oplus \mathbb{R}^2)$ with $b < 0$ have $\mathscr{D}_U$ a ball.
(c) If $\mathscr{D}_U$ is not a ball, then it is a half space (bounded by an affine hyperplane) or the exterior of the closure of a ball.

□