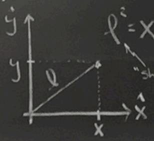
physics: identify a target invariant, build transformations that support it.

<u>example</u>:



$\ell^2 = x^2 + y^2$

"shouldn't" change

define simple (but general) <u>linear</u> transformation:

$\bar{x} = Ax + By \; , \; \bar{y} = Fx + Gy$

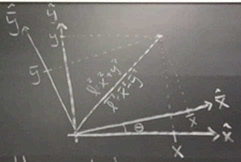
for const.s $\{A, B, F, G\}$, w/ shared origin. Demand:

$$\bar{\ell}^2 \equiv \bar{x}^2 + \bar{y}^2 = x^2 + y^2 = \ell^2$$

so that len.s are unchanged.

$$\bar{x}^2+\bar{y}^2 = (A^2+F^2)x^2+2xy(AB+FG)+(B^2+G^2)y^2$$
$$= x^2+y^2 \text{ gives 3 eqn.s.}$$

$$A^2+F^2=1,\quad AB+FG=\varnothing,\quad B^2+G^2=1$$



we are left w/ 1 free parameter, "θ"

$A=\cos\theta$, $F=-\sin\theta$, $B=\sin\theta$, $G=\cos\theta$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$ rotation of axes through an angle θ

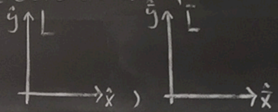
the relation can be inverted: $\theta \longrightarrow -\theta$.

$$\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}$$

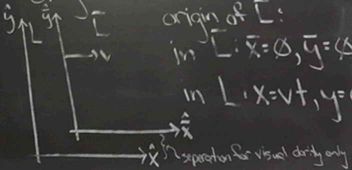
are there other "invariants" of interest? "Special relativity" provides one. We need some setup to see it.

=

2 sets of axes, L & $\bar{L}$:



$\bar{L}$ travels to the right, through L, w/ constant speed v. At $t=\emptyset$, the origins coincide.



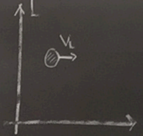
origin of $\bar{L}$:

in $\bar{L}$: $\bar{x}=\emptyset$, $\bar{y}=\emptyset$

in L: $x=vt$, $y=\emptyset$

Experiment 1

at $t = \emptyset$, as origins coincide, throw a ball w/ speed v_L relative to L:



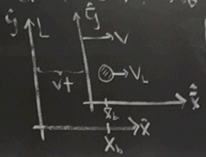
? at time t - where is the ball in L?

$x_b = v_L t$

speed of ball relative to $\bar{L}$:

$v_{\bar{L}} = v_L - v$ (check: if $v_L = v$, $v_{\bar{L}} = \emptyset$ ✓)

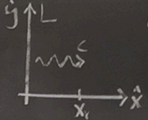
? where is the ball in $\bar{L}$ at time t? $\bar{x}_b = v_{\bar{L}} t = (v_L - v)t$



(as drawn here, $v_L > v$.)

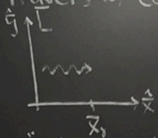
Experiment 2

at $t = \emptyset$, as origins coincide, throw a "photon" (turn on a flashlight pointing along the shared $\hat{x}, \hat{\bar{x}}$ axis) at speed c.



At time t, photon is at $x_\gamma = ct$, travels at constant speed c relative to L.

the observation, in $\bar{L}$, is that the photon also travels w/ constant speed c... relative to $\bar{L}$!



$$\bar{x}_\gamma = c\,\bar{t}$$

time, as measured in $\bar{L}$

1. "velocity addition" must change
2. L + $\bar{L}$ measure time differently.

comparing the "photons" location in L, $\bar{L}$:

$$x_\gamma = ct \Rightarrow x_\gamma^2 - (ct)^2 = 0$$

$$\bar{x}_\gamma = c\bar{t} \Rightarrow \bar{x}_\gamma^2 - (c\bar{t})^2 = 0$$

$$\bar{x}_\gamma^2 - (c\bar{t})^2 = x_\gamma^2 - (ct)^2$$

idea - this holds $\forall\ (x, ct)$ to $(\bar{x}, c\bar{t})$

(clearly true for c, but why at other speeds?)

$$\boxed{\bar{x}^2 - (c\bar{t})^2 = x^2 - (ct)^2}$$

a quadratic, similar in form to $\bar{x}^2 + \bar{y}^2 = x^2 + y^2$

except for that sign ...

Transformation

What relation between $\bar{x}, c\bar{t}$ & x, ct preserves this quantity?

$$\bar{x} = Ax + B(ct)$$
$$c\bar{t} = Fx + G(ct)$$

again start w/ a linear relation, leaving the x-ct origin fixed

demand: $\bar{x}^2 - (c\bar{t})^2 = x^2 - (ct)^2$:

$$\bar{x}^2 - (c\bar{t})^2 = (A^2 - F^2)x^2 + 2xct(AB - FG) + (B^2 - G^2)(ct)^2 \overset{!}{=} x^2 - (ct)^2$$

gives, again, 3 eqn.s in 4 "unknowns"

$$A^2 - F^2 = 1, \quad AB - FG = \emptyset, \quad G^2 - B^2 = 1.$$

$$A = \cosh\eta, \quad F = \sinh\eta, \quad G = \cosh\eta, \quad B = \sinh\eta$$

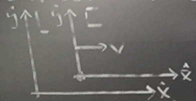
$$\begin{pmatrix} c\bar{t} \\ \bar{x} \end{pmatrix} = \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

w/ inverse given by $\eta \to -\eta$

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \cosh\eta & -\sinh\eta \\ -\sinh\eta & \cosh\eta \end{pmatrix} \begin{pmatrix} c\bar{t} \\ \bar{x} \end{pmatrix}$$

the "Lorentz" transformation

how is η related to this picture?



pick a point, any point

origin of $\bar{L}$: $\bar{x} = \emptyset$ $(\bar{y} = \emptyset)$ in $\bar{L}$

$x = vt$ $(y = \emptyset)$ in L

$$\bar{x} = x\cosh\eta + ct\sinh\eta$$

$$\emptyset = vt\cosh\eta + ct\sinh\eta \Rightarrow \boxed{\tanh\eta = -\frac{v}{c}}$$

from $\tanh\eta = -\frac{v}{c}$, and
$\cosh^2\eta - \sinh^2\eta = 1$, we
learn that:

$$\cosh\eta = \frac{1}{\sqrt{1-v^2/c^2}} \equiv \gamma$$

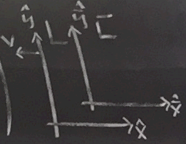
$$\sinh\eta = \frac{-v/c}{\sqrt{1-v^2/c^2}} = -\frac{v}{c}\gamma$$

η is the "rapidity"
the Lorentz transformation is:

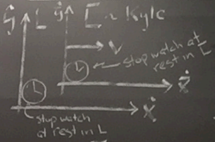
$$\begin{pmatrix} c\bar{t} \\ \bar{x} \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{v}{c}\gamma \\ -\frac{v}{c}\gamma & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

w/ inverse
obtained via
$v \longrightarrow -v$:

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \gamma & \frac{v}{c}\gamma \\ \frac{v}{c}\gamma & \gamma \end{pmatrix} \begin{pmatrix} c\bar{t} \\ \bar{x} \end{pmatrix}$$



Time Dilation



origins coincide,
synch.:
$t = 0,\ \bar{t} = 0$

time $\bar{t}$ elapses on Kyle's stopwatch - how much time has passed on the clock at rest in L?

Lorentz: $ct = \gamma c\bar{t} + \frac{v}{c}\gamma\bar{x}$ ← $\bar{x} = 0$ is the location of the clock in $\bar{L}$

$$t = \frac{1}{\sqrt{1 - v^2/c^2}}\,\bar{t} > \bar{t}$$

At Kyle's normal pace: $v = \frac{4}{5}c$

$t = \frac{5}{3}\bar{t}$ ← $\left.\begin{array}{l}\text{for 1 year in } \bar{L},\\ 5/3 \approx 1.7 \text{ years in } L\end{array}\right\}$ faster is younger

Proper Time

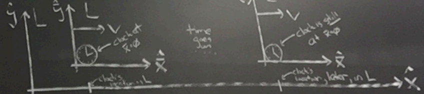
the invariant:

$$\bar{x}^2 - (c\bar{t})^2 = x^2 - (ct)^2$$

holds for infinitesimal intervals.

$$d\bar{x}^2 - c^2 d\bar{t}^2 = dx^2 - c^2 dt^2$$

a clock is at rest at the origin of $\bar{L}$



in $\bar{L}$, the clock is at rest, $d\bar{x}=0$ and ticks off time $d\bar{t}$

in L, the clock is <u>moving</u>, $dx \neq 0$, and ticks off time dt.

the intervals are related:

$$d\bar{x}^2 - c^2 d\bar{t}^2 = dx^2 - c^2 dt^2$$

or $c^2 d\bar{t}^2 = c^2 dt^2 - dx^2$

the clock in $\bar{\Sigma}$ records the "propre" (own) time. It is at rest in $\bar{\Sigma}$, at least instantaneously. Proper time typically denoted τ.

Parametrizing Motion

common, in physics, to use time as a parameter to describe motion, $x(t)$

Could we instead use τ? from

$$c^2 d\tau^2 = c^2 dt^2 - dx^2$$

if we use τ to param., $x(\tau)$, then: $dx = \frac{dx}{d\tau} d\tau$, and we must also parametrize the "movement" in time, $t(\tau)$, w/ $dt = \frac{dt}{d\tau} d\tau$, and

$$c^2 d\tau^2 = \left[c^2 \left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dx}{d\tau}\right)^2\right] d\tau^2 \quad \text{or}$$

$$1 = \left(\frac{dt}{d\tau}\right)^2 - \frac{1}{c^2}\left(\frac{dx}{d\tau}\right)^2$$

an affine parametrization for $V_\tau \doteq \begin{pmatrix} dt/d\tau \\ dx/d\tau \end{pmatrix}$

we can still use t, $x(t)$ has:

$$dx = \frac{dx}{dt} dt$$

familiar t-param'ed velocity component

then

$$c^2 d\tau^2 = c^2 dt^2 - \left(\frac{dx}{dt}\right)^2 dt^2$$

or

$$dt^2 = \frac{d\tau^2}{\left(\sqrt{1 - \frac{\left(\frac{dx}{dt}\right)^2}{c^2}}\right)^2}$$

our time-dilation result from earlier.