

# DIY hyperbolic geometry

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written for Mathcamp 2015

## **Abstract and guide to the reader:**

This is a set of notes from a 5-day Do-It-Yourself (or perhaps Discover-It-Yourself) introduction to hyperbolic geometry. Everything from geodesics to Gauss-Bonnet, starting with a combinatorial/polyhedral approach that assumes no knowledge of differential geometry. Although these are set up as “Day 1” through “Day 5”, there is certainly enough material hinted at here to make a five *week* course. In any case, one should be sure to leave ample room for play and discovery before moving from one section to the next.

Most importantly, these notes were meant to be acted upon: if you’re reading this without building, drawing, and exploring, you’re doing it wrong!

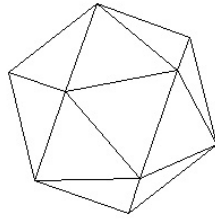
**Feedback:** As often happens, these notes were typed out rather more hastily than I intended! I would be quite happy to hear suggestions/comments and know about typos or omissions. For the moment, you can reach me at [kpmann@math.berkeley.edu](mailto:kpmann@math.berkeley.edu)

As some of the figures in this work are pulled from published work (remember, this is just a class handout!), please only use and distribute as you would do so with your own class materials.

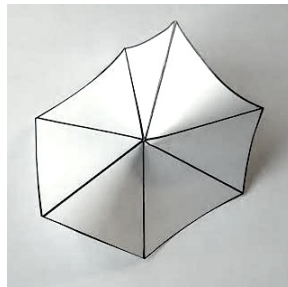
## Day 1: Wrinkly paper

If we glue equilateral triangles together, 6 around a vertex, and keep going forever, we build a flat (Euclidean) plane. This space is called  $\mathbb{E}^2$ .

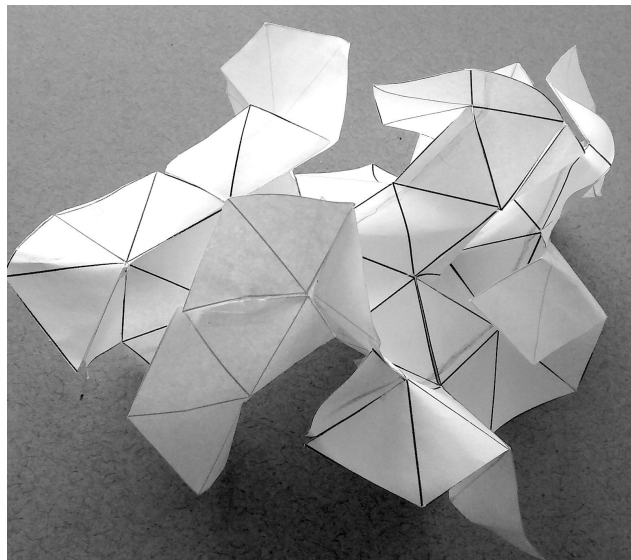
Gluing 5 around a vertex eventually closes up and gives an icosahedron, which I would like you to think of as a *polyhedral approximation* of a round sphere.



Gluing seven triangles around a single vertex gives a shape that is also curved, but not like a sphere.



Much like the 6-around a vertex Euclidean plane, we can continue to glue 7 triangles around each vertex forever, building an infinite space. This space is the *hyperbolic plane*, also called  $\mathbb{H}^2$ . (Technically what we actually build with triangles is a polyhedral approximation of  $\mathbb{H}^2$  – the real  $\mathbb{H}^2$  is a smoothed-out version, like the sphere is in comparison with the icosahedron). Here's a picture of a piece of it:



Our goal for this week is to explore this space, first using paper models like these, then through the more standard *Poincaré disc* and *half-space* models.

## Geodesics

A *geodesic* is often defined to be the shortest distance between two points. A better definition of geodesic is a *straight line*, although it takes some time to make sense of what straight should be in a curved space.

On a folded piece of paper, geodesics are folded straight lines. The same applies to the icosahedron, and to our hyperbolic space models – to draw a geodesic starting in a given direction, draw a straight line on a triangle, and when you get to the edge, flatten the edge momentarily and continue the straight line on the other side.

**Exercise 1** (Optional). Show that the geodesics on a sphere are exactly the great circles by looking at straight lines on better and better polyhedral approximations of the sphere. (The approximations in the figure below are given by successively subdividing each triangular face into 4 equilateral triangles.)



**Exercise 2** (Geodesics on  $\mathbb{H}^2$ ). Draw some geodesics on your paper model of  $\mathbb{H}^2$ . What do they look like? More specifically:

1. Draw two geodesics which start out as parallel lines, but eventually diverge
2. Draw a geodesic that contains a segment parallel to, and very close to, the edge of one of the equilateral triangle pieces. Notice how it follows a path made up of edges. We'll call these special geodesics *edge-path geodesics*. Try to describe all possible edge-path geodesics. What paths of edges can they follow?
3. Draw a BIG triangle on  $\mathbb{H}^2$  with all sides geodesics.
4. Can you draw a (big) geodesic sided rectangle? (i.e. four sides, 90 degree corners). Drawing it inside a single flat triangle piece is cheating! If you can't, can you *prove* that it's impossible to do so? Hint/further question: What is the angle sum of your big geodesic triangle?

We'll return to study more properties of geodesic triangles later on. If you're interested more generally in figuring out the right way to think about *straight lines on curved spaces*, I highly recommend the first chapters of Henderson's book *Experiencing Geometry* [2].

## Area and circumference of discs

Consider the Euclidean plane  $\mathbb{E}^2$  tiled by unit side length triangles. We can estimate the area of a disc of radius  $r$  by counting the number of triangles in it. Since the area of a triangle is a constant (say  $\lambda$ ) and a hexagon-worth of triangles fills out a fixed proportion of the disc (say  $\kappa$ ), our count of the number of triangles gives  $\frac{\kappa}{\lambda}$  times the area of the disc. If we're just interested in understanding what area looks like as a function of  $r$  (is it linear? quadratic? polynomial?), we don't even need to worry about what  $\kappa$  and  $\lambda$  are.

We can similarly estimate the circumference of the disc using the circumference of the hexagon. We'll call this the *combinatorial* area and length.

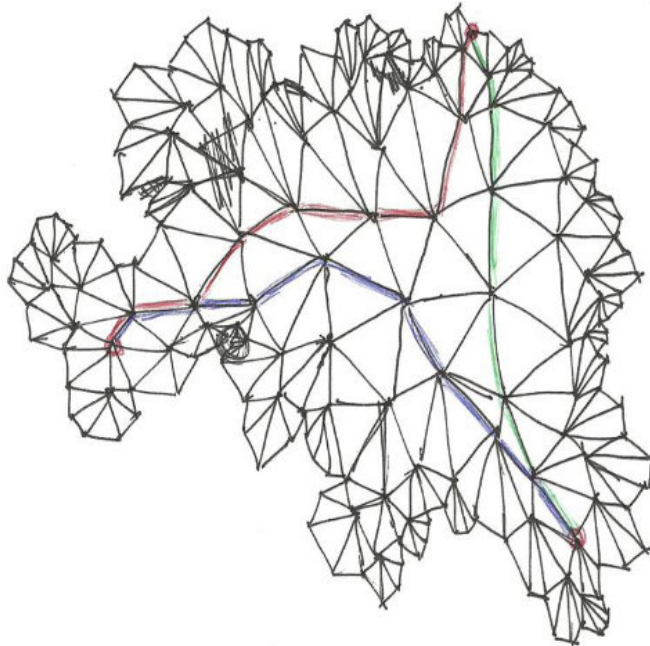


Figure 1: An edge-path geodesic triangle (pretty much). Drawing by D. Studenmund

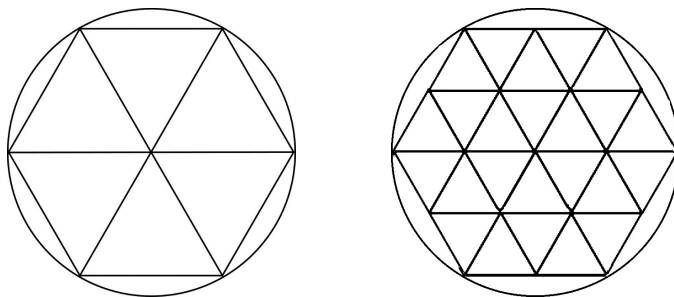


Figure 2: a “radius  $r$  hexagon” fills out a fixed proportion of a disc of radius  $r$ .

**Definition 3.** The *combinatorial area* of a region in a triangulated space is the number of triangles in that region. The *combinatorial length* of a curve is the number of edges of triangles that make up that curve.

In  $\mathbb{H}^2$ , things are a little different than in  $\mathbb{E}^2$ . Adding a 1-triangle thick annulus around our starting heptagon gives something that doesn’t quite look like a bigger heptagon (let’s call it a *triangulated disc* of radius 2) and our “fixed proportion of the area” argument doesn’t work here. (Exercise: why not?). Still, counting the number of triangles in such a triangle-disc still gives a reasonable first estimate of the area of a disc of radius 2. Adding more annular rings to build a bigger triangle-disc gives a reasonable approximation of discs of larger radii. See exercise 5 below.

**Exercise 4.** (Area and circumference in  $\mathbb{E}^2$  and  $\mathbb{H}^2$ )

1. Find a formula for the combinatorial area and circumference of a hexagonal “disc” of radius  $r$  in  $\mathbb{E}^2$ .

2. Find an estimate for the combinatorial circumference of a triangulated disc of radius  $r$  in  $\mathbb{H}^2$ . (estimate = some reasonable upper and lower bounds. Hint: try to bound below by something like  $2^r$ ).
3. Find a formula or estimate for the combinatorial area of a triangulated disc of radius  $r$  in  $\mathbb{H}^2$ . It may help to draw a flat picture of a space with 7 triangles around a vertex (a *regular degree 7 graph*) like in Figure 1 or use the one on the last page of these notes.

**Exercise 5.** (optional) Assume our equilateral triangles have side length 1. Which points on the circumference of the triangulated disc of radius  $r$  in  $\mathbb{H}^2$  are distance (very close to)  $r$  from the center along edge-path geodesics? Why does this make our triangulated disc a good approximation of the genuine disc of radius  $r$  in  $\mathbb{H}^2$ ?

## Notes

The polyhedral paper model of hyperbolic space was popularized by (and perhaps even invented by?) W. Thurston. Although too advanced for our purposes here, he has a wonderful book *Three-Dimensional Geometry and Topology* [4] that begins with a DIY-style introduction to  $\mathbb{H}^2$ .

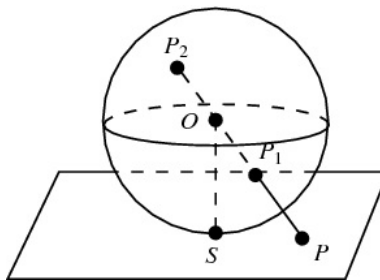
## Day 2: Mapping the hyperbolic plane

### Three projections of the sphere

**Question 6.** Why are all maps of the earth imperfect in some way (distorting area, or angles, or distances)? In other words, why are all images of a sphere, or a part of a sphere, on a flat plane distorted?

Many people answer this question with something like “because geodesics are curved (great circles) on the sphere, and straight lines on the plane. Curved things can’t be made straight” However, there is a way to represent the round earth – actually, half of it – on a flat plane, where the image of each geodesic is a straight line.

#### 1. Gnomonic projection (preserves straight lines)



Gnomonic projection, from <http://mathworld.wolfram.com/StereographicProjection.html>

The gnomonic projection maps the lower hemisphere onto  $\mathbb{E}^2$  (covering the whole plane).

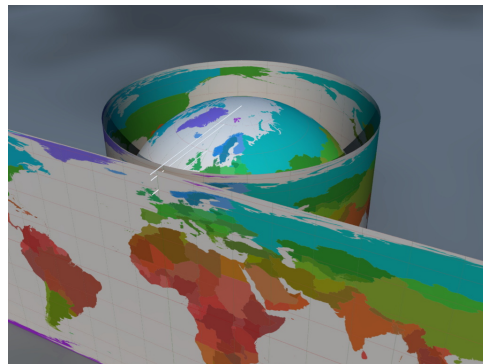
**Theorem 7.** The image of a great circle under the gnomonic projection is a straight line

*Proof.* Each flat plane passing through the center of the sphere intersects the sphere along a great circle. Conversely, each great circle can be described as the intersection of a plane through  $C$  with the sphere. The rays defining the gnomonic projection stay on this plane, which then intersects  $\mathbb{E}^2$  in a straight line.  $\square$



Although geodesics under gnomonic projection are straight lines, the image of the sphere is quite badly distorted – notice how area is badly distorted at points far away from the point of tangency to the plane. If we want to preserve area, we can use cylindrical projection instead.

## 2. Cylindrical projection (preserves area)



Cylindrical projection, wikimedia commons

**Theorem 8** (Archimedes). The area of any region on the sphere does not change under cylindrical projection.

However, distances and angles can be very badly distorted – look what happened to Greenland in the picture. Moreover, though some great circles are mapped to straight lines under cylindrical projection (exercise – which ones?), not all are.

**3. Stereographic projection** (preserves angles). Stereographic projection, shown below, preserves angles and sends small circles to circles. In fact, almost all circles are sent to circles...

**Exercise 9.** Describe the image of great circles under stereographic projection.

**Exercise 10** (optional, challenging). Prove that cylindrical projection preserves area, and prove that stereographic projection preserves angles and sends circles to circles.

## Two projections of $\mathbb{H}^2$

**1. The Poincaré disc.** If you looked at the last page of the notes in order to do one of the counting exercises in the last section, you have already seen a picture of the Poincaré disc model

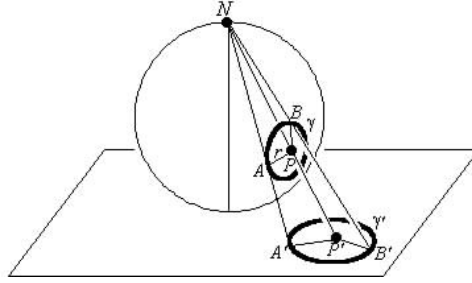
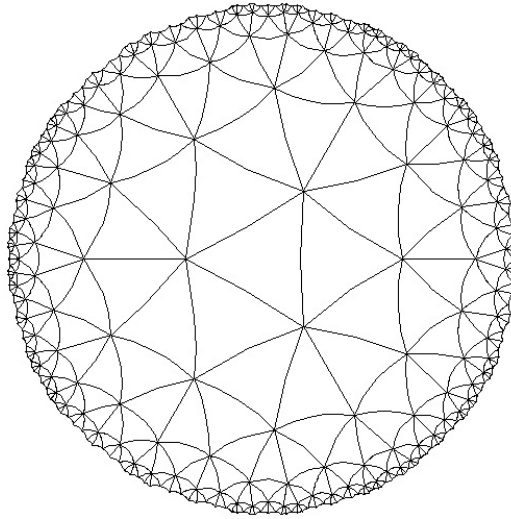


Figure 3: From D. Henderson’s *Experiencing Geometry* [2]

of hyperbolic space – it’s that picture of “seven triangles around a vertex” – an infinite, regular, degree 7 graph, drawn so as to fit inside a round disc. But in order to fit inside the disc, the area of triangles has to be more and more distorted. (Can you estimate *how* distorted, using your estimate on area from Exercise 4?)



**Exercise 11.** (Geodesics in the disc)

1. Draw the image of some geodesic segments on the Poincaré disc by copying them off of your paper model.
2. Can you characterize what all edge-path geodesics look like?
3. What does this tell you about straight lines in the Poincaré disc model? Does this remind you of any of the projections of the sphere?

The Poincaré disc also lets us talk about a smooth version of hyperbolic space. Think of the disc as a uniform space, symmetric under any rotation about the center point (your paper model has 7-fold rotational symmetry at a vertex, and 3-fold about a point in the middle of a triangle). The disc is also symmetric under translations – just as in  $\mathbb{E}^2$ , there is a translation taking any point to any other, and preserving distances and angles, in  $\mathbb{H}^2$  there is also an *isometry* taking any point to any other.

You can find some movies of isometries made by C. Goodman-Strauss here: <http://comp.uark.edu/~strauss/hyperbolia/gallery2.html>

You can build a (almost) smooth paper model by gluing annuli together as in the instructions here. <http://www.math.cornell.edu/~dwh/books/eg00/supplements/AHPmodel/>. (This website also has instructions on how to crochet a model of  $\mathbb{H}^2$ !)



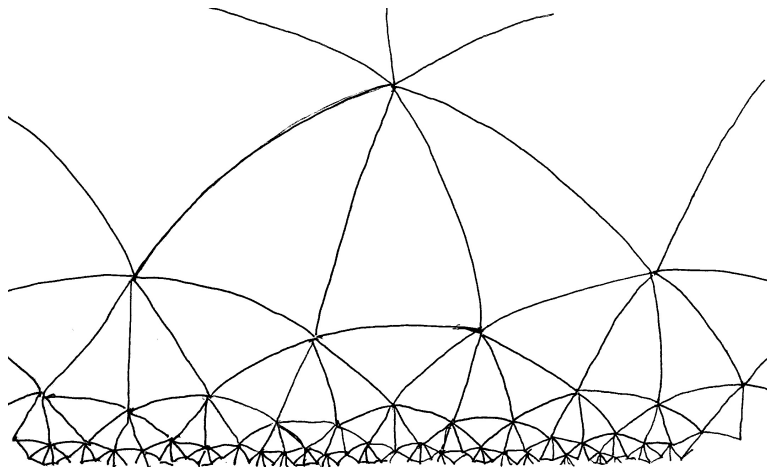
A paper model of the hyperbolic plane, constructed from annuli. <http://theiff.org/current/events/hyperbolic-surfaces/>

To see that every point of  $\mathbb{H}^2$  is like every other, cut out a small piece of the annuli model surface, and slide it around over the surface. Just as a small cap of a sphere of radius  $R$  can be slid smoothly to any point on the same sphere and then rotated by any angle, and a small piece of flat plane can be slid and rotated around the plane, the same is true of  $\mathbb{H}^2$ .

The geodesics you drew on the Poincaré disc based on your triangulated hyperbolic space are *not quite* the same as geodesics on the smooth version, but are pretty close and give you a general idea of how they behave. On the smooth version, geodesics are semi-circles that meet the boundary at right angles.

## 2. The upper half space model

Imagine taking a Poincaré disc of bigger and bigger radius. The boundary starts to look more and more straight as the disc grows. You can imagine this process “limiting” onto a half-plane. We think of the half-plane as an infinite-area disc with “circle” boundary a straight line. (There is a good way to make this idea precise using stereographic projection!) Here is a picture of the 7-triangles around a vertex tiling on a piece of the half-plane.



The benefit of this “map projection” of hyperbolic space is that it’s easy to describe the *hyperbolic metric* – in other words, what *scale* our map is at different places. If you draw a



short line segment on the plane (say of length  $l$ ) passing through a point with  $y$ -coordinate  $y_0$ , it will represent a path of length approximately  $y_0 l$  in  $\mathbb{H}^2$ . In other words, distances at height  $y$  are distorted by a factor of  $y$ .

If you know about complex numbers and the complex plane, you can write down an easy formula for going between the disc and the upper half space models. Think of the disc as  $\{z \in \mathbb{C} : |z| < 1\}$  and the upper half plane as  $\{z \in \mathbb{C} : z = a + bi, \text{ and } a > 0\}$ . Then the transformation

$$f(z) = \frac{z - i}{z + i}$$

maps the half-plane onto the disc. (And its inverse  $f^{-1}(z) = \frac{-iz - i}{z - 1}$  maps the disc to the half-plane).

A function of the form  $f(z) = \frac{az+b}{cz+d}$ , where  $a, b, c$  and  $d$  are complex numbers, is called a *Möbius transformation*. These have the property that the image of circles are circles... provided that, as in stereographic projection, we consider a straight line as a kind of circle.

**Exercise 12.** Using the fact that Möbius transformations map circles to circles *and* preserve angles, describe the geodesics of  $\mathbb{H}^2$  in the upper-half plane model.

Complex numbers can also be used to give a precise formula for distance. Any two points  $z$  and  $w$  in the upper half plane can be connected by a unique *orthocircle* – a vertical line or half-circle meeting the  $x$ -axis at right angles. In fact, these orthocircles are the images of *geodesics* in this model. To find the distance between  $z$  and  $w$ , first find the orthocircle through  $z$  and  $w$ , and name its endpoints  $A$  and  $B$  so that  $A, z, w, B$  reads in order along the path. (We let  $A$  or  $B$  be infinity if its a vertical line.) Then

$$d(z, w) = \ln \frac{(w - A)(z - B)}{(A - z)(B - w)}$$

This fraction always turns out to be a real number greater than 1, so its logarithm is positive, and in fact  $d(z, w)$  gives a *metric*, a good notion of distance. (Of course, you need to check symmetry – not hard – and the triangle inequality – rather tricky!).

**Exercise 13.** 1. Using this distance formula, check that a small line segment at height  $y$  has length distorted by a factor of (approximately)  $y$ .

2. Show that length is *additive* along vertical lines: if  $z_1, z_2$  and  $z_3$  are three points, in order, on the  $y$ -axis, then  $d(z_1, z_2) + d(z_2, z_3) = d(z_1, z_3)$ .
3. To go further, read about the *cross ratio* in the book *The Four Pillars of Geometry* by J. Stillwell [3]. Then explain why hyperbolic distance is additive along orthocircles, not just vertical lines.

### 3. The Klein model.

Like the gnomonic projection of the earth, there is a projection of  $\mathbb{H}^2$  onto the plane where geodesics are straight lines. It's called the Klein (disc) model. Figure 4 is a picture comparing the three different projections.

That such a model exists is very special – the following theorem is a special case of a result proved by Beltrami.

**Theorem 14.** (Beltrami's theorem) Suppose  $X$  is a space that has a projection to  $\mathbb{R}^2$  where the image of each geodesic in  $X$  is a straight line. Then  $X$  is either a part of the plane, part of a sphere, or part of  $\mathbb{H}^2$ .

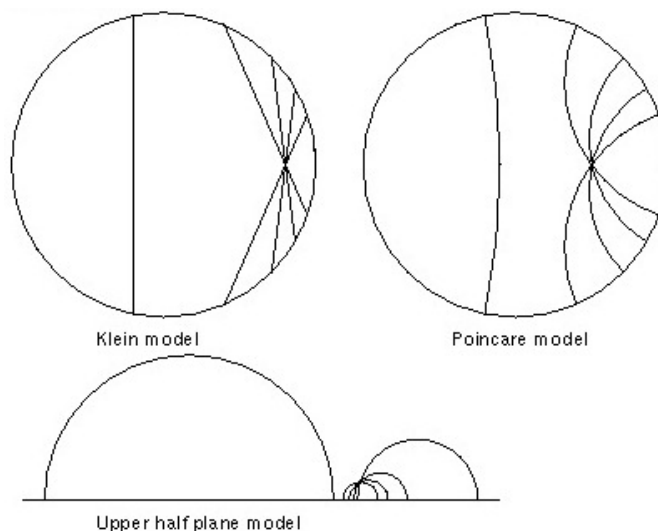


Figure 4: Comparing projections of  $\mathbb{H}^2$ . From [www.geom.uiuc.edu/docs/forum/hype/model.html](http://www.geom.uiuc.edu/docs/forum/hype/model.html)

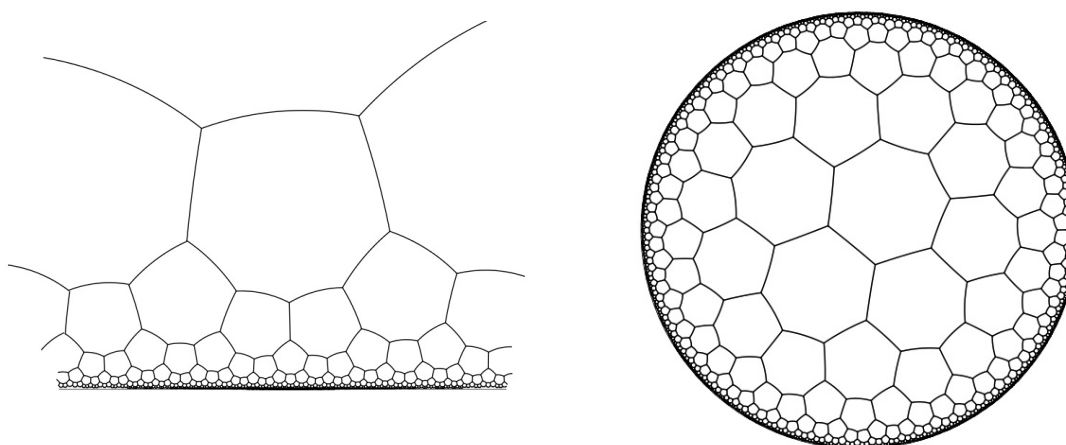


Figure 5: tiling of half space and disc by regular heptagons

## Visualizing hyperbolic space in the different models through tilings

M.C. Escher drew dozens of tilings of Euclidean space, but only four tilings of the Poincaré disc. These hyperbolic tilings were extremely labor intensive—without computers (or even a calculator), it is remarkably difficult to produce a precise regular tiling of the Poincaré disc. Escher had some mathematical help from geometer H.S.M. Coxeter, but had to do the drawing on his own with standard drafting tools (e.g. ruler and compass).

Jos Leys, using computers, has recently transformed a number of Escher's tilings of the plane into tilings of  $\mathbb{H}^2$ . See [http://www.josleys.com/show\\_gallery.php?galid=325](http://www.josleys.com/show_gallery.php?galid=325) for many beautiful examples on both the Poincaré disc and upper half plane model.

**Exercise 15** (optional, challenge). Can you figure out what Leys' method is (not the technical details, just the general strategy) for turning a tiling of the Euclidean plane into a tiling of the



Figure 6: tiling of half space and disc by regular lizards (Jos Leys, after M.C. Escher.)

What symmetries of  $\mathbb{H}^2$  are visible here?

hyperbolic plane?

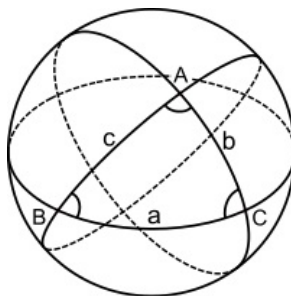
We'll look more at tilings and symmetries on Day 4.

### Day 3: Hyperbolic and spherical triangles, *curvature*

#### The area of a triangle on a sphere

A *lune* is a wedge of a sphere, cut out by two great circles. Since the total area of a sphere of radius  $R$  is  $4\pi R^2$ , the area of a lune of angle  $\alpha$  is  $4\pi R^2(\alpha/2\pi) = 2\alpha R^2$ .

One can calculate the area of a spherical triangle by examining how the three pairs of lunes determined by the pairs of adjacent sides of a triangle cover the sphere, with some overlap. It's best to do this by drawing on a physical sphere (orange, grapefruit, tennis ball...)



**Exercise 16.** (Spherical triangles)

1. What is a formula for a triangle with angles  $\alpha, \beta, \gamma$  on a sphere of radius  $R$ ?
2. What angle sums are possible?

## The area of a triangle on $\mathbb{H}^2$

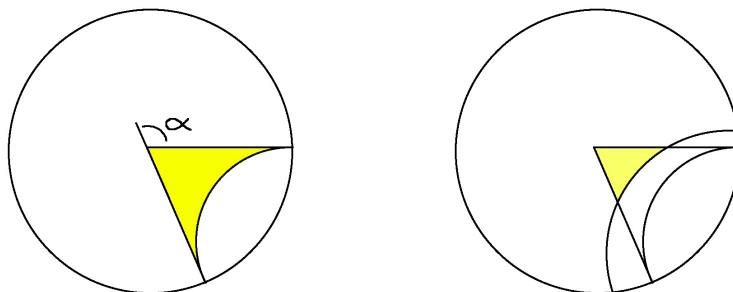
We use a similar method to derive the formula for a triangle on  $\mathbb{H}^2$ . This strategy is taken from Chapter 7 of [2].

We'll need to use one fact about the hyperbolic plane that we discussed yesterday: every point is exactly like every other. Precisely, given any two points, there is an *isometry* of  $\mathbb{H}^2$ , analogous to a translation of the plane, that takes one point to the other. And given any single point, there is an isometry that rotates around that point. (This is easiest to see in the Poincaré disc when you rotate around the center of the disc).

First we define a special region of hyperbolic space that will play the role of the lune.

**Definition 17.** A *one-angle triangle* (OAT) of *exterior angle*  $\alpha$  is the region in  $\mathbb{H}^2$  cut out by two geodesic rays forming an angle of  $\pi - \alpha$  at a point, together with a third infinite geodesic connecting their endpoints.

On the left of Figure is a picture of such a region on the Poincaré disc:



(a) An OAT of exterior angle  $\alpha$       (b) as a limit of finite triangles...

Although this seems to make perfect sense on our picture of the disc and on the upper half plane, the “endpoints” of the rays are actually infinite distance away from the center. Try thinking about what this region looks like on your wrinkly paper! To make this definition absolutely precise, we should really talk about the OAT as a *limit* of finite regions as on the right of the figure above.

The fact that there is an isometry of  $\mathbb{H}^2$  taking any point and direction to any other means that every OAT of angle  $\alpha$  is isometric to any other OAT of exterior angle  $\alpha$ . In particular, they all have the same area. Let  $A(\alpha)$  denote the area of an OAT of exterior angle  $\alpha$

**Exercise 18.** Show that  $A$  is additive, i.e. that  $A(\alpha) + A(\beta) = A(\alpha + \beta)$ .

Since  $A$  is continuous (in fact, all you really need is that  $A$  is monotone – a smaller external angle gives a smaller area), and additive, it must be *linear*. So there is some constant  $\lambda$  so that  $A(\alpha) = \lambda\alpha$ .

When  $\alpha = \pi$ , we get a “triangle” with all angles equal to 0. This is called an *ideal triangle*, and has all vertices on the boundary of the disc.

**Exercise 19.** Derive a formula for a hyperbolic triangle with angles  $a, b, c$  in terms of  $\lambda$ . (a big hint is given in Figure 9 below.) Note that in particular, you just proved that the angles of a triangle determine the area!

You'll notice that you also showed that a hyperbolic triangle always has angle *less than* 180 degrees, and that the maximum possible area is  $\pi\lambda$ , which occurs when the triangle is an *ideal triangle*.

**Exercise 20.** Derive a formula for the area of a polygonal region with geodesic sides (on the unit sphere, and in  $\mathbb{H}^2$ ) in terms of the angles at the vertices, by dividing the region up into triangles.

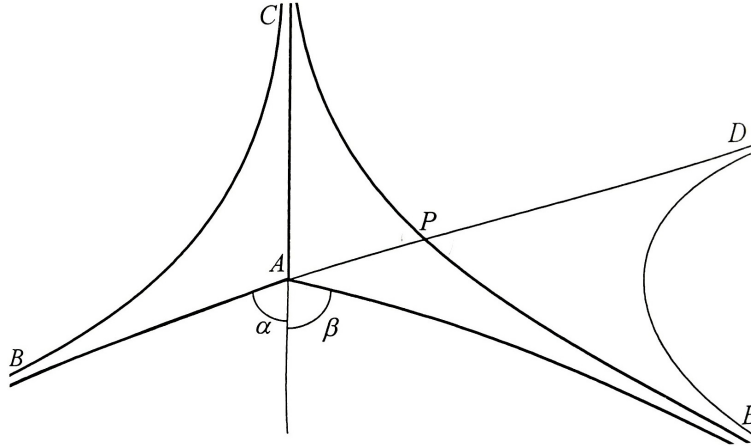


Figure 8: From [2], a hint for additivity of  $A(\alpha)$

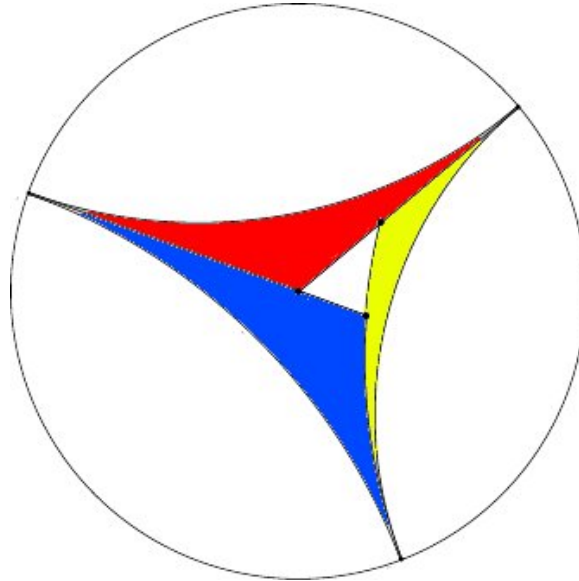


Figure 9: Hint for exercise 19

## The Gauss-Bonnet formula

Just as the formula for area of triangles (or polygons) on the sphere depended on the radius, the formula on  $\mathbb{H}^2$  depended on  $\lambda$ . Like spheres, hyperbolic spaces come with different degrees of curvyness, and our constant  $\lambda$  is a measure of how “curved” the space is. The area formula we derived is a special case of a famous theorem:

**Theorem 21** (Gauss-Bonnet for polygons). For any polygon  $P$  on a sphere, in  $\mathbb{E}^2$  or on  $\mathbb{H}^2$  with geodesic sides, we have

$$\text{area}(P) = \frac{1}{K}(2\pi - \text{sum of external angles})$$

where  $K$  is the *curvature* of the space.

The flat plane  $\mathbb{E}^2$  has curvature 0, a sphere of radius  $R$  has curvature  $1/R^2$  (bigger spheres are less curved), and the hyperbolic plane has *negative curvature*. The description of the metric on the Poincaré disc given above is for a space with curvature = -1. It is possible to calculate the approximate curvature of the model you made out of equilateral triangles (just like you can say what the “radius” of an icosahedron is), and this depends on the size of the triangles that you are working with.

Some spaces cannot be assigned a single number for curvature – they are more or less curvy in different places. Instead, one assigns a different number to each point in the space, called the *Gaussian curvature* at that point. The Gaussian curvature is negative if there is “too much area” in the immediate region around the point (such spaces are locally saddle-shaped), positive if there is “too little area” (like on the sphere), and zero if the immediate area around the point is isometric to a flat plane (flat, conical, or cylindrical).



A surface with non-constant curvature



Even worse!

**Exercise:** Which points on these spaces have positive (respectively, negative) curvature?

The Gauss-Bonnet formula generalizes to these spaces too, but you need to measure the average curvature over a region by taking an integral...

### Further notes

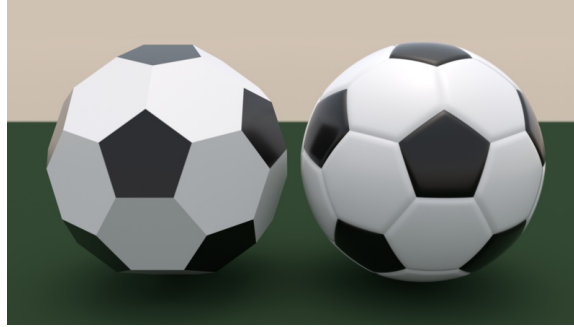
All of high school trigonometry (cosine law, sine law, special properties for right triangles, etc.) can be done on the sphere and in hyperbolic space, the trigonometric identities just take a slightly different form. See Chapter 6 in [2] for an introduction to congruencies of triangles, and [4] for more hyperbolic trigonometry.

## Day 4: Tilings, isometries, and hyperbolic soccer

Today we explore the many different ways to tile the hyperbolic plane. We’ll discuss a number of different strategies, with the goal of a) making some art like M.C. Escher, and b) understanding (groups of) symmetries of  $\mathbb{H}^2$ .

### Hyperbolic soccer

We can build a “rounder” polygonal approximation of the sphere by slicing off each corner of the icosahedron. The result is a *truncated icosahedron*, the original triangle sides are now hexagons and the sliced off vertices form new pentagonal sides. This shape is also known by the technical term *soccer ball*.



Let's do the same with our tiling of  $\mathbb{H}^2$ . What do we get?

**Exercise 22.** Build a hyperbolic soccer ball plane. You can find detailed instructions and a template here: [http://theiff.org/images/IFF\\_HypSoccerBall.pdf](http://theiff.org/images/IFF_HypSoccerBall.pdf)

**Exercise 23.** (Challenge: What is the shape of a hyperbolic soccer ball?) Is it possible to make a real *ball* with heptagons and hexagons in the hyperbolic soccer pattern? Is it possible to make any closed shape? If so, what would it look like?

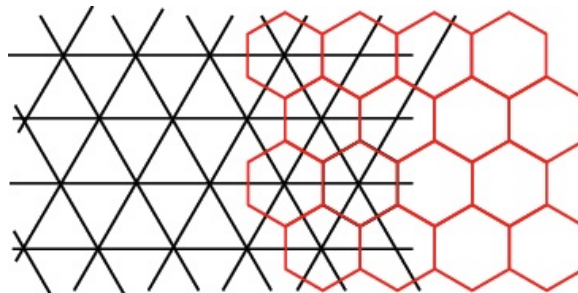
**Exercise 24.** (Version of the exercise above, with more guidance) Use an Euler characteristic argument to show that you can't make cover a ball with the hyperbolic soccer pattern. In fact, any closed surface tiled by heptagons and hexagons, with three faces meeting at every vertex, must have negative Euler characteristic (unless there are no heptagons, in which case you can have Euler characteristic zero).

Addendum to exercise: I had some fantastic students come up with a plan for a hyperbolic genus 2 soccer surface. Can you do it?

## Dual polyhedra and dual tilings

Another way to construct a new tiling is to take the *dual* of an existing one.

**Definition 25.** The dual of a tiling is constructed by taking the center of each tile as a vertex and joining the centers of adjacent tiles. The dual of a polyhedron is constructed similarly, by taking the center of each face to be a vertex. (In fact, it's exactly the same construction if you consider a polyhedron as a "tiling" of the sphere)



from <http://mathworld.wolfram.com/DualTessellation.html>

**Exercise 26.** Show that the dual of the dual of a polyhedron (or tiling) is the original polyhedron (tiling).

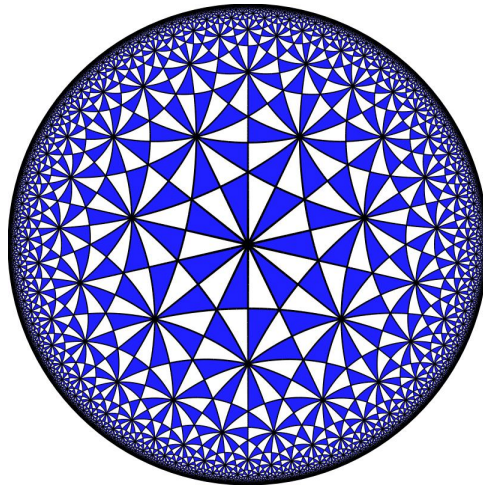
**Exercise 27.** Prove that every symmetry of a polyhedron is also a symmetry of its dual polyhedron. Prove that every symmetry of a tiling of the plane is also a symmetry of its dual. Show the same is also true for the hyperbolic plane. (this should help you understand isometries of  $\mathbb{H}^2$ )

**Exercise 28.** (optional, assumes you know some group theory.) Describe the *groups of symmetries* of each of the Platonic solids. Which have the same symmetry group? Which are duals of each other?

### Which shapes tile the hyperbolic plane?

**i) Triangles and regular tilings.** Take a geodesic triangle with angles  $\pi/p, \pi/q$  and  $\pi/r$  for some integers  $p, q, r$ . Reflect it in one of its sides, this gives another triangle with the same angles, lined up with the first. If you keep going, you'll eventually get  $2p$  triangles around the vertex of angle  $\pi/p$ , (and  $2q$  around the  $\pi/q$  vertex, and  $2r$  around the  $\pi/r$  vertex). If you keep going forever, you eventually fill all of hyperbolic space.

This is easiest to see if you start by running the same kind of experiment with a  $90 - 45 - 45$  triangle in Euclidean space. The tiling of  $\mathbb{E}^2$  that you get is pretty boring, but in hyperbolic space there are many more options, because there exist triangles with *any* angle sum of  $< 180$  degrees. Here's an example of a  $\pi/2, \pi/3, \pi/7$  triangle tiling.



**Exercise 29.** What  $\pi/p, \pi/q, \pi/r$  triangles exist in  $\mathbb{E}^2$ ? What tilings do they give? What about in  $\mathbb{H}^2$ ? Can you visualize what these tilings look like?

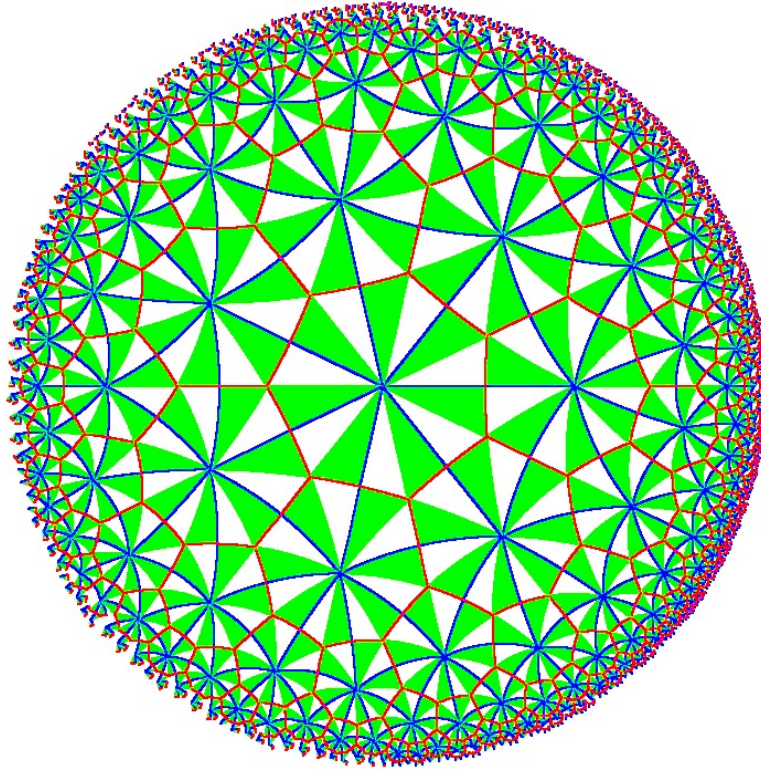
If you take two adjacent triangles and merge them into 1, this shape also tiles the plane, and the tiles are not reflected but *rotated* and *translated* copies of each other. You can even take more than two – below is an example of tilings created out of 14 triangles each (outlined in red, the other in blue) – one making up heptagons, and the other large triangles. Look familiar?

Using a similar strategy, you can show that you can tile hyperbolic space with *any* regular polygon, and in many different ways.

**Exercise 30** (Tiling with regular polygons). . Define a *regular polygon* to be a polygon with all sides equal length, and all angles equal.

1. Show that you can tile the hyperbolic plane with regular 4-gons (squares), with 5 around each vertex.
2. Show that you can tile the hyperbolic plane with regular 4-gons, with  $n$  around each vertex, for any  $n \geq 5$





3. Show that you can tile the plane with regular  $n$ -gons. How many must you put around each vertex (you have options). In each case, what must the interior angle of the  $n$ -gon be?

**Exercise 31** (Symmetry groups, optional). If you know something about *groups of symmetries*, or in particular about *reflection groups* in  $\mathbb{E}^2$ , you can try to generalize this to  $\mathbb{H}^2$ . What can you say about reflection groups in  $\mathbb{H}^2$ ? Can you get *new, different* groups? Start by trying to understand the group created by reflections in the sides of the  $\pi/2, \pi/3, \pi/7$  triangle. Stillwell's book [3] is a good introductory resource on this.

**ii) Semi-regular tilings** Escher's *Circle limit 3* is a tiling based on squares and triangles. You can make a polygonal approximation of this by taking cutting out squares and triangles from paper, and taping them together according to Escher's pattern.

To create another semi-regular tiling of hyperbolic space, pick a finite collection of regular polygons, and find a way to glue them together so that there is more than 360 degrees of angle at each vertex. You also want there to be some pattern to the gluing – it should be invariant under translations in two different directions, so that there is a network of points scattered throughout the space and a symmetry (“translation”) taking each point to each other. (Such a network of points is called a *lattice*). The result is a polyhedral approximation of  $\mathbb{H}^2$ , with a semi-regular tiling.

For an easy example, you can modify Escher's fish pattern to consist of pentagons and triangles. Figure 11 shows a picture of the result, created by D. Dunham. For more examples, see [1].

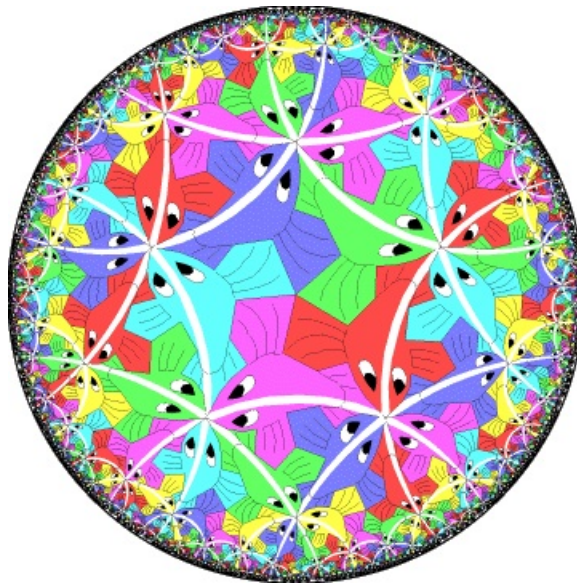
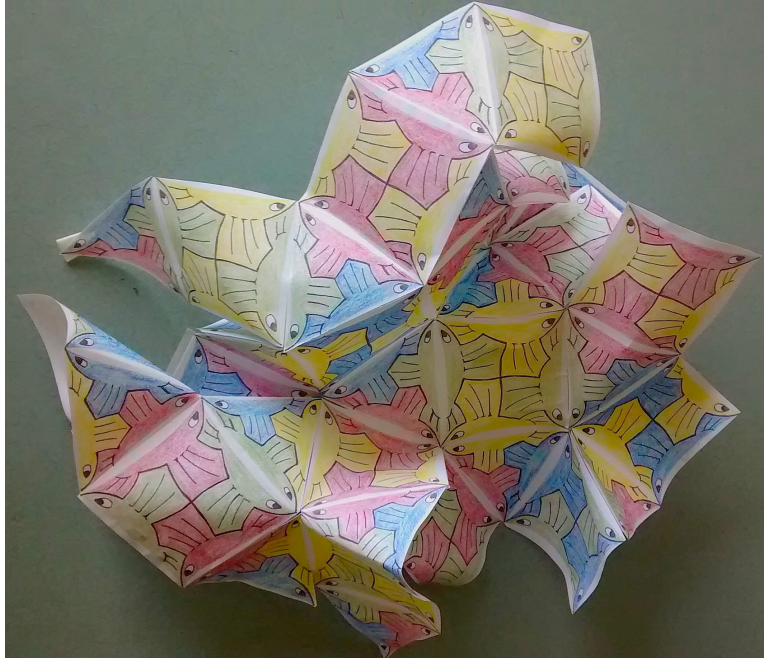


Figure 11: A semi-regular tiling of  $\mathbb{H}^2$  decorated with fish. Image by D. Dunham [1]

### How to tessellate be M. C. Escher

Now that we've tiled  $\mathbb{H}^2$  with polygons, we can modify these to produce tilings by more interesting shapes (lizards, fish,...). The general recipe is as follows:

1. Take a tiling by polygons, and think deeply about the symmetries of this tiling.
2. Imagine translating one of the polygons to one of its copies, in a way that realizes a symmetry of the tiling. Which sides of the original polygon land on which sides of the copy?

- Using this information, remove a piece of one side of your polygon and glue it onto another, puzzle-piece style. Voilà, a new tiling, by a more interesting shape. (warning: you might have to remove-glue from the same side!)

**Exercise 32.** Explore! Create some original tilings of hyperbolic space!

## Day 5: Potpourri

Most of today will be devoted to catching up and working on exercises from the previous days, but here are a couple extra tidbits, just for fun. The first is about tiling Euclidean space, the second a foray into the 3-dimensional world, and the third a discussion of sports in the hyperbolic plane...

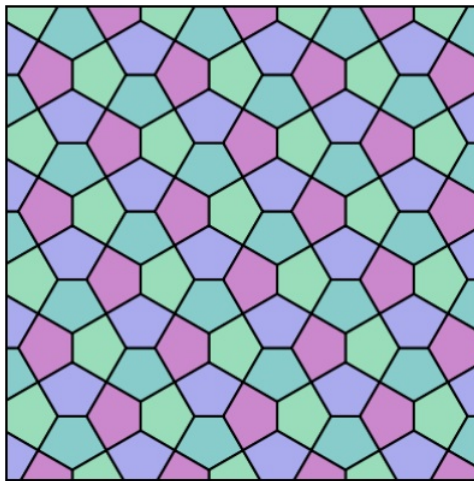
### i) You can't tile the plane with heptagons

We start with a warm-up theorem:

**Theorem 33.** You can't tile the plane with regular heptagons

*Proof.* Glue regular heptagons together. If you put 2 at each vertex, you don't have enough angle— to make a flat plane, you need 360 degrees of surface around any point. If you put 3 or more at a vertex, you have too much angle – in fact, you just built hyperbolic space!  $\square$

A similar strategy to the proof above lets you show that you can't tile the plane with regular pentagons. But you *can* tile the plane with convex pentagons of a different shape. In fact, there are many ways to do so! (see [https://en.wikipedia.org/wiki/Pentagon\\_tiling](https://en.wikipedia.org/wiki/Pentagon_tiling))



Amazingly, this doesn't work at all for heptagons. And the reason is an area/circumference counting game like we did on day 1.

**Theorem 34.** There is no collection of finitely many 7-sided, convex tiles that can be used to tile the plane.

*Proof – outline.* Suppose that you had a collection of such tiles and built a tiling of the plane. Look at a very large, round-ish region of the plane formed by a bunch of your 7-sided tiles. If you pick a large enough, round enough region, the area of this region will be roughly proportional

to the square of the perimeter. (This is just using the fact that the area of a disc is  $\pi r^2$ , and the circumference  $2\pi r$ ).

Now we'll use a counting argument to get a contradiction. The tiling of this region satisfies Euler's formula

$$V - E + F = 1$$

where  $V$  is the total number of vertices,  $E$  the number of edges, and  $F$  the number of faces, or tiles. Since each tile has 7 edges, counting the total number of edges gives

$$7F = 2(\# \text{ interior edges}) + (\# \text{ exterior edges})$$

Let  $P$  be the number of edges around the perimeter. Then the formula above becomes

$$7F = 2(E - P) + P = 2E - P.$$

Since the tiles are convex, each interior vertex has at least 3 incident edges. Also, each vertex on the boundary has at least two incident edges. Counting vertices gives

$$2E \geq 3(\# \text{ interior vertices}) + 2(\# \text{ exterior vertices}) = 3(V - P) + 2P = 3V - P$$

Now solve the equations above for  $P$  in terms of  $F$ . This gives

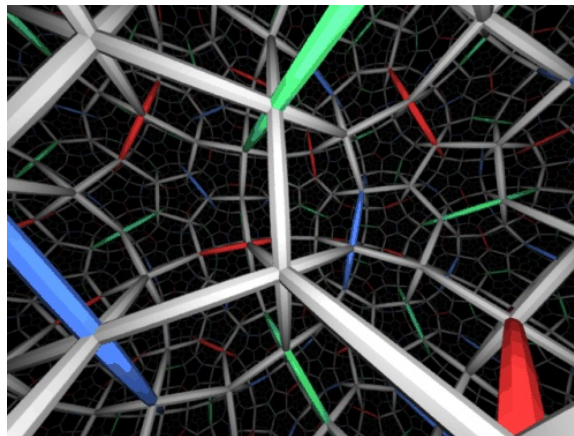
$$F \leq P + 6.$$

Since we are using only finitely many kinds of tiles,  $F$  is proportional to the area of the region (each tile has area between  $c_1$  and  $c_2$ , so  $F$  is between  $c_1$  times the area and  $c_2$  times the area); similarly,  $P$  is proportional to the perimeter. This contradicts our estimate above that the area  $F$  is supposed to be roughly equal to  $P^2$ .  $\square$

## ii) 3-dimensional hyperbolic space

If you glue together a bunch of cubes of the same size, 2 meeting at each face, and 8 at each vertex, you build a 3-dimensional block. If you do this forever in all directions, you've build 3-dimensional Euclidean space,  $\mathbb{E}^3$ , tiled by cubes.

What if you glue together solids with bigger dihedral angles, or glue more cubes around a vertex? You guessed it – the result is 3-dimensional hyperbolic space.



A tiling of 3-dimensional hyperbolic space by regular dodecahedra. From the *geometry center*

This movie <http://www.geom.uiuc.edu/video/sos/> explores 3-dimensional hyperbolic (and Euclidean) spaces, motivated by the question *What if our universe is hyperbolic?*. It's based on an excellent book by J. Weeks, called *The Shape of Space* [5].

You might also enjoy “not knot” about 3-dimensional hyperbolic shapes. <http://www.geom.uiuc.edu/video/>

### iii) Just for fun

In this video (<https://youtu.be/u6Got0X41pY>) Dick Canary reviews hyperbolic geometry and discusses what it might be like to play some sports in hyperbolic space. (spoiler: hyperbolic sports are hard!) Can you invent a sport well-adapted to playing in  $\mathbb{H}^2$ ?

Appendix: pictures too big to fit in the text

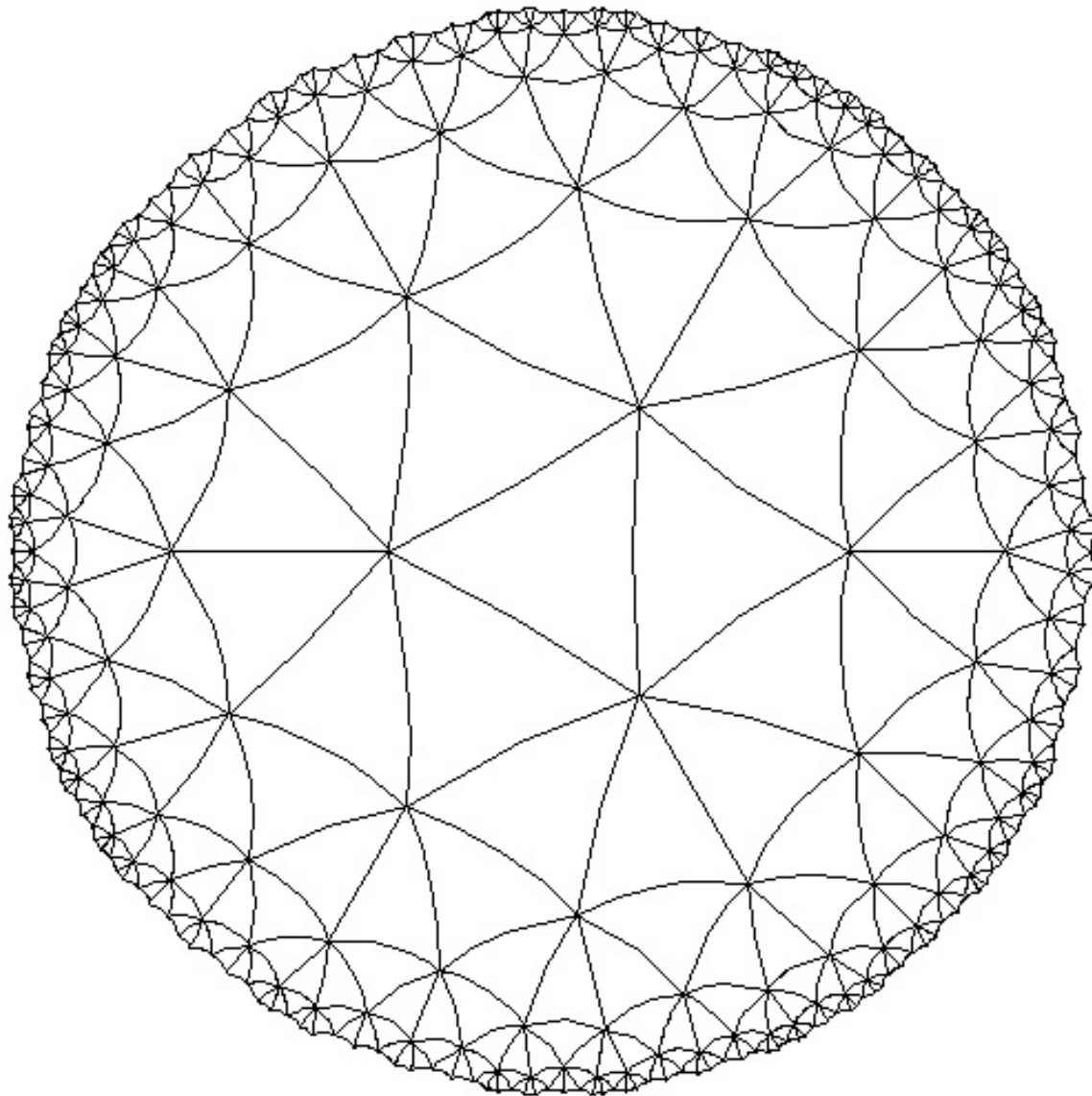
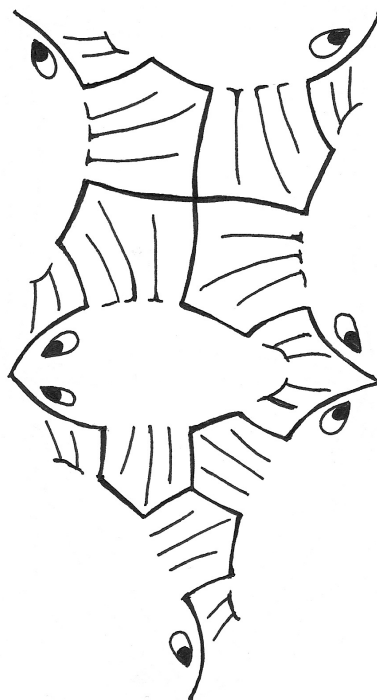
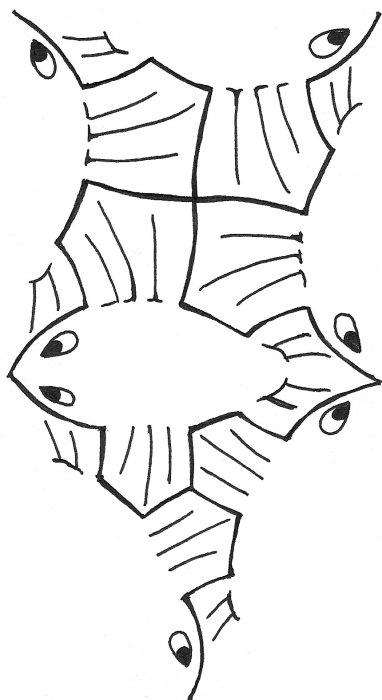
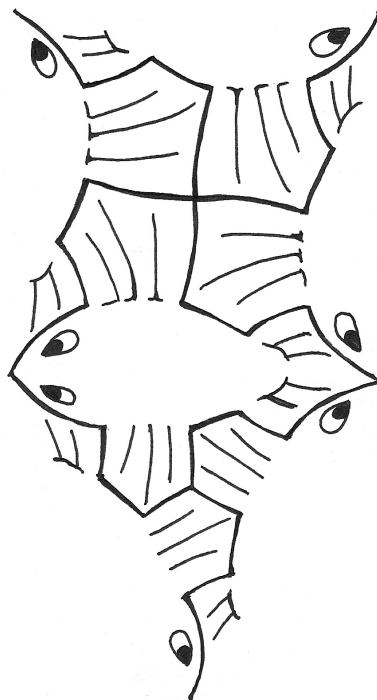
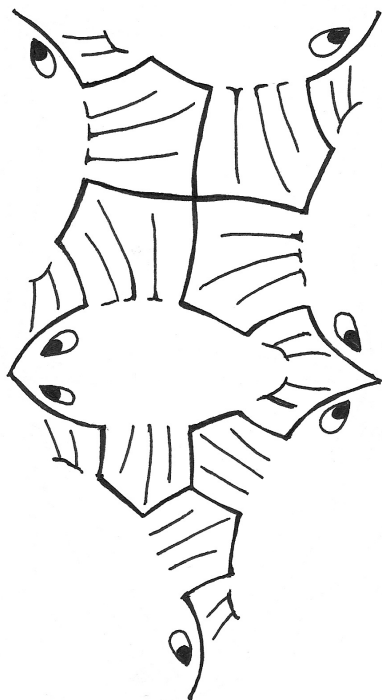


Figure 12: 7 triangles around a vertex.

Draw on me!

### Template for making a paper version of Circle Limit III

Instructions: make many copies, cut-out, color (careful – what colors are adjacent?), and tape together.



## References

- [1] D. Dunham, *Transformation of Hyperbolic Escher Patterns* <http://www.d.umn.edu/~ddunham/isis4/index.html>
- [2] D. W. Henderson, D Taimina, *Experiencing Geometry: In Euclidean, Spherical, and Hyperbolic Spaces*. Prentice Hall, 2001.
- [3] J. Stillwell, *The Four Pillars of Geometry*. Springer-Verlag, New York, 2005
- [4] W. Thurston, *Three-Dimensional Geometry and Topology, Volume 1*, Princeton University Press, 1997.
- [5] J. Weeks, *The shape of space*. CRC Press, 2001.

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