# Lecture 8

Friday, February 6, 2015 9:58 AM

For $\varphi: G \longrightarrow H$ defined $\Sigma(\varphi) = G/K$, $K = \ker(\varphi)$

$= \{ X_a = \varphi^{-1}(a) \mid X_a \neq \emptyset \}$

$= \{ gK \mid g \in G \}$

$= \{ Kg \mid g \in G \}$

$X_a \cdot X_b = X_{ab}$

$\underbrace{gK \cdot g'K = (gg')K}$

$gK$ is called a <u>left</u> <u>coset</u>, $gK = \{ gk \mid k \in K \}$

$Kg$ — " — right coset.

Know $gK = \varphi^{-1}(a) \Longrightarrow \varphi(g) = a$

$g'K = \varphi^{-1}(b) \Rightarrow \varphi(g') = b$

Does $X_{ab} \stackrel{?}{=} (gg')K$ ? $\varphi(gg') = \varphi(g)\varphi(g') = ab$ ✓

$\overset{\shortparallel}{\varphi^{-1}(ab)}$

<u>Prop</u> Let $N \trianglelefteq G$. Then $\{ gN \mid g \in G \}$ forms a partition of $G$. I.e. $\forall u, v \in G$,

$uN = vN \iff v^{-1}uN = N$

$uN = vN \iff u, v$ are representatives of the same coset.

Pf Clearly $G = \bigcup_{g \in G} gN$. If $x \in gN \cap g'N$ then $x = gn = g'n'$ for $n, n' \in N$.

$$\Rightarrow g = g'\underbrace{n'n^{-1}}_{\in N}.$$ For any $t \in N$,

$$gt = (g'n'n^{-1})t = g'\underbrace{((n'n^{-1})t)}_{\in N} \in g'N$$

$\Rightarrow gN \subseteq g'N$.

Swap $gN, g'N \Rightarrow g'N \subseteq gN$. $\square$

Defn $N \subseteq G$ is a normal subgroup of $G$ if $N \leq G$ and $gNg^{-1} = N \ \forall g \in G$.

Write $N \trianglelefteq G$.

Prop For $N \leq G$, then

① $uN \cdot vN = (uv)N$ is well-defined iff $N \trianglelefteq G$.

② If the op above is well-defined, then it makes $\{gN \mid gN\} = G/N$ into a group w/ $(gN)^{-1} = g^{-1}N$, $1_{G/N} = N$.

<u>Pf</u> For ①, first assume well-def'n. Then if $u, u' \in uN$, $v, v' \in vN$, then $uvN = u'v'N$. For any $g \in G$, $n \in N$, let $u = 1$, $u' = n$, $v = v' = g^{-1}$. Then $1 \cdot g^{-1} N = ng^{-1} N$

$$g^{-1}N = ng^{-1}N \implies ng^{-1} \in g^{-1}N$$

$$\implies \exists n' \in N \text{ s.t. } ng^{-1} = g^{-1}n'$$

$$\implies gng^{-1} = n' \in N$$

$$\implies gNg^{-1} \subseteq N$$

$$\iff gNg^{-1} = N.$$

Now assume $N \trianglelefteq G$. For $u, u' \in uN$, $v, v' \in vN$ have $u' = un$, $v' = vm$ for $n, m \in N$. Then

$$u'v' = unvm = uvv^{-1}nvm$$

$$= uv\underbrace{\underbrace{(v^{-1}nv)}_{\in N}m}_{\in N}$$

by normality!

$u'v' = uvn'$ for $n' = v^{-1}nvm \in N$

By partition prop, get $uvN = u'v'N$. ② ✓

**Thm** For $N \leq G$, TFAE:

① $N \trianglelefteq G$

② $N_G(N) = G$

③ $gN = Ng \ \forall g \in G$

④ $uN \cdot vN = uvN$ makes $G/N$ into a group

⑤ $gNg^{-1} \subseteq N \ \forall g \in G$

⑥ $N$ is the kernel of some gp hom $\varphi$ with domain $G$.

Recall $N_G(N) = \{ g \in G \mid gng^{-1} \in N \ \forall n \in N \}$

Pf ④ $\Rightarrow$ ⑥: Define the natural projection

$$\pi : G \longrightarrow G/N$$
$$g \longmapsto gN$$

It's a hom: $\pi(gh) = (gh)N \overset{④}{=} gN \cdot hN$.

$$\ker(\pi) = \{ g \in G \mid \pi(g) = N \}$$
$$= \{ g \in G \mid gN = N \}$$
$$= \{ g \in G \mid g \in N \} = N. \quad \square$$

Why 5? Assume $gNg^{-1} \subseteq N \quad \forall g \in G$.

Set $g = h^{-1}$ for some $h \in G$.

Get $h^{-1}N(h^{-1})^{-1} \subseteq N$

$$h^{-1}Nh \subseteq N$$

$$Nh \subseteq hN$$

$$N \subseteq hNh^{-1}.$$

$h$ was arbitrary, so we get the opp inc.!

e.g. $\mathbb{R}/\mathbb{Z} = S^1$

$$\mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}_n \qquad \ker(\varphi) = n\mathbb{Z}$$

$$\Rightarrow \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}.$$

⚠ Not everything is normal:

$$H = \langle (1\ 2) \rangle \leq S_3$$

$$\{1, (1\ 2)\}$$

$$(2\ 3)H = \{(2\ 3), (1\ 3\ 2)\}$$

$$H(2\ 3) = \{(2\ 3), (1\ 2\ 3)\}$$

$\neq$