Lecture 5

Monday, February 2, 2015 9:59 AM

Homomorphisms:

Notes ① gps G, H always have trivial hom

$$1 : G \longrightarrow H \qquad 1(gh) = 1$$
$$g \longmapsto 1 \qquad 1(g)\cdot 1(h) = 1\cdot 1 = 1$$

② If $\varphi : G \to H$ is a hom, observe that

$$\varphi(1) = \varphi(1\cdot 1) = \varphi(1)\cdot\varphi(1) \implies 1 = \varphi(1).$$

③ $1 = \varphi(1) = \varphi(xx^{-1}) = \varphi(x)\varphi(x^{-1})$

mult on left by $(\varphi(x))^{-1}$ to get

$$(\varphi(x))^{-1} = \varphi(x^{-1}). \quad \forall x \in G.$$

Group Actions

Defn A (left) group action of G on a set A, denoted $G \circlearrowright A$, is a map $G \times A \longrightarrow A$ satisfying

$$(g, a) \longmapsto g\cdot a$$

① $g_1\cdot(g_2\cdot a) = (g_1 g_2)\cdot a \quad \forall g_1, g_2 \in G, a \in A$

② $1\cdot a = a \quad \forall a \in A.$

The data of $G \times A \to A$ a gp action is equivalent to a homomorphism $G \longrightarrow S_A$ called the permutation representation of $G \circlearrowright A$, $\varphi_{G \circlearrowright A}$.

$$g \longmapsto \begin{array}{c} A \\ \downarrow \sigma_g \\ A \end{array} \qquad \begin{array}{c} a \\ \downarrow \\ g\cdot a \end{array}$$

First check that $\sigma_{gh} = \sigma_g \circ \sigma_h$:

$$\sigma_{gh}(a) = (gh)\cdot a$$
$$= g\cdot(h\cdot a) = g\cdot \sigma_h(a) = \sigma_g(\sigma_h(a)).$$

Note that $id_A = \sigma_1 = \sigma_{gg^{-1}} = \sigma_g \circ \sigma_{g^{-1}} \Rightarrow \sigma_g$ is a bijection $A \to A$.

$$\| $$
$$\sigma_{g^{-1}g} = \sigma_{g^{-1}} \circ \sigma_g$$

Given a hom $\varphi: G \longrightarrow S_A$ we get $G \circlearrowright A$ via

$$g\cdot a = (\varphi(g))(a).$$

<u>Exc</u> Show that these processes reverse each other.

<u>e.g.</u> ⓪ Any gp G acts on a set A <u>trivially</u> :

$$g\cdot a = a$$

The assoc perm rep'n is the trivial hom

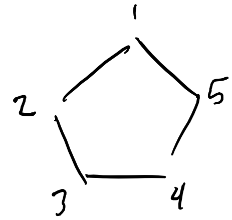
$$1: G \longrightarrow S_A.$$

① The identity homomorphism $S_A \longrightarrow S_A$ has assoc action $\sigma\cdot a = \sigma(a)$

$$S_A \times A \longrightarrow A$$
$$(\sigma, a) \longmapsto \sigma\cdot a = \sigma(a)$$

② $D_{2n} \circlearrowright \underline{n} = \{1, 2, \dots, n\}$

$n = 5$:



Action records how the labels are permuted.

③ $G \circlearrowright G$ via left multiplication:

$$g \cdot h = gh.$$

<u>Cayley's Theorem</u> The gp hom $G \xrightarrow{\varphi} S_G$ assoc with the left mult action is injective.

In particular, every gp is isomorphic to a subgp of a symmetric gp & every group of order $n < \infty$ is isomorphic to a subgroup of S_n.

<u>Pf</u> Suffices to show $\ker(\varphi) = 1 = \{1\}$.

Suppose $g \in \ker(\varphi)$ i.e. $\varphi(g) = id : G \longrightarrow G$.

Thus $1 = (\varphi(g))(1) = g \cdot 1 = g$ and $\ker \varphi = 1$. $\square$

<u>Subgroups</u> $H \subseteq G$ is a <u>subgroup</u> of G if

① $H \neq \emptyset$

② H is closed under mult

③ H is closed under inverses.

<u>Notation</u> Write $H \leq G$ when H is a subgroup of G.

<u>Etc</u> $\leq$ is transitive!

e.g.
- G, $1 \leq G$
- F a field then $\{\pm 1\} \leq F^{\times} = F \setminus \{0\}$ w/ mult
- $\{1, r, r^2, r^3, \ldots, r^{n-1}\} \leq D_{2n}$

Subgroup Criterion $H \subseteq G$ is a subgroup of G iff
1. $H \neq \emptyset$
2. $x, y \in H \Rightarrow xy^{-1} \in H$.

If $|H| < \infty$, then it suffices to check $H \neq \emptyset$ & closed under mult.

Pf $H \leq G \Rightarrow$ 1 + 2 ✓.

Assume 1 + 2. By 1, we have some $x \in H$ whence by 2, $x \cdot x^{-1} = 1 \in H$. Apply 2 to $1, x \in H$ to get $1 \cdot x^{-1} = x^{-1} \in H$. Now for $x, y \in H$, now know $y^{-1} \in H$ so 2 $\Rightarrow x \cdot (y^{-1})^{-1} = x \cdot y \in H$. Thus $H \leq G$. □

Subgps from group actions

$G \circlearrowright S \ni s$. Define the stabilizer of s (isotropy of s) to be $G_s = \{g \in G \mid g \cdot s = s\} = \mathrm{Stab}_G(s)$.

Prop $G_s \leq G$.

Pf $1 \in G_s$ b/c $1 \cdot s = s$. so $G_s \neq \emptyset$.

For $x \in G_s$ have
$$s = 1 \cdot s = (x^{-1}x)\cdot s = x^{-1}\cdot(x\cdot s) = x^{-1}\cdot s \implies x^{-1} \in G_s.$$

For $x, y \in G_s$ have
$$(xy)\cdot s = x\cdot(y\cdot s) = x \cdot s = s. \qquad \square$$

Defn The **kernel** of $G \circlearrowright S$ is $\ker(G \circlearrowright S) = \bigcap_{s\in S} G_s$.

Note $\bigcap_{s \in S} G_s = \{g \in G \mid g\cdot s = s \ \forall s \in S\} \leq G$.

Alternately, $\ker(G \circlearrowright S) = \ker(\varphi_{G \circlearrowright S}) \leq G$.

Exc