Lecture 49

Tuesday, April 28, 2015 10:07 AM

Linear map $T : V \longrightarrow V$, V a fin dim F-v.s.

Suppose $c_T(x)$ has all its roots in F.

Then $c_T(x) = \prod (x - \lambda_i)^{k_i}$

$k_i \in \mathbb{Z}^+$, $\lambda_i \in F$

Thus all elementary divisors of $F[x]$-module V_T are of the form $(x-\lambda)^k$

Study $F[x] / ((x-\lambda)^k)$:

$$(\bar{x}-\lambda)^{k-1}, \quad (\bar{x}-\lambda)^{k-2}, \quad \ldots, \quad \bar{x}-\lambda, \quad 1$$

$$(\bar{x}-\lambda)^j = \sum_{i=0}^{j} \binom{j}{i} (-\lambda)^{j-i} \bar{x}^i$$

$$= (-\lambda)^j + j(-\lambda)^{j-1}\bar{x} + \binom{j}{2}(-\lambda)^{j-2}\bar{x}^2 + \cdots + \bar{x}^j$$

so the matrix expressing $(\bar{x}-\lambda)^j$'s in terms of $\cancel{1, \bar{x}, \bar{x}^2, \ldots, \bar{x}^{k-1}}$ $\bar{x}^{k-1}, \bar{x}^{k-2}, \ldots, 1$

is of the form $\begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ * & & & 1 \end{pmatrix}$ w/ det 1

$\Rightarrow (\bar{x}-\lambda)^{k-1}, (\bar{x}-\lambda)^{k-2}, \ldots, 1$ is a basis!

How does x act on this basis?

$$x = \lambda + (x-\lambda)$$

$$(\bar{x}-\lambda)^{k-1} \xmapsto{x\cdot} \lambda(\bar{x}-\lambda)^{k-1} + (\bar{x}-\lambda)^k = \lambda(\bar{x}-\lambda)^{k-1}$$

$$(\bar{x}-\lambda)^{k-2} \xmapsto{x\cdot} \lambda(\bar{x}-\lambda)^{k-2} + (\bar{x}-\lambda)^{k-1}$$

$$\vdots$$

$$(\bar{x}-\lambda)^{j} \xmapsto{x\cdot} \lambda(\bar{x}-\lambda)^{j} + (\bar{x}-\lambda)^{j+1}$$

The matrix for $x\cdot$ on $F[x]/((x-\lambda)^k)$ wrt chosen basis is

$$\begin{pmatrix} \lambda & 1 & & & 0 \\ & \lambda & 1 & & \\ & & \lambda & \ddots & \\ 0 & & & \ddots & 1 \\ & & & & \lambda \end{pmatrix} =: \text{Jordan block of size } k \text{ w/ eigenvalue } \lambda.$$

Elementary divisor decomposition of V_T:

$$V_T \cong \bigoplus_{i=1}^{t} F[x] / \left((x-\lambda_i)^{k_i} \right)$$

Can choose a basis s.t. the matrix for T is of the form

$$\begin{pmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_t \end{pmatrix}$$

where J_i is the Jordan block of size k_i w/ eigenvalue λ_i.

This the <u>Jordan</u> <u>canonical</u> <u>form</u> of T.

<u>Thm</u> If the eigenvalues of T are all in F, then $\exists$ basis of V s.t. T is in JCF wrt this basis, and this form is unique up to permutation of the Jordan blocks. $\square$

Cor ① If a matrix A is similar to a diagonal matrix D, then D is the Jordan canonical form of A (so no Jordan blocks of size > 1).

② Two diagonal matrices are similar $\iff$ diagonal entries are the same up to permutation. $\square$

Cor Matrix A w/ all eigenvalues in F.

Then A is similar to a diagonal matrix over F

$\iff$ Jordan blocks all have size 1

$\iff$ elementary divisors are of the form $x-\lambda$

$\iff$ $m_A(x)$ has no repeated roots.

Pf Most of this is obvious.

Min poly $m_A(x)$ = least common multiple of min polys of Jordan blocks

$\implies$ Each elementary divisor is $(x-\lambda)^1$. $\square$

e.g. Suppose we know the invariant factors of T

$$(x-1)(x-3)^3, \quad (x-1)(x-2)(x-3)^3, \quad (x-1)(x-2)^2(x-3)^3$$

The elementary divisors of T are

$$x-1,\ (x-3)^3, \quad x-1,\ x-2,\ (x-3)^3, \quad x-1,\ (x-2)^2,\ (x-3)^3$$

$$\begin{pmatrix}
\boxed{1} & & & & & & & \\
& \boxed{1} & & & & & & \\
& & \boxed{1} & & & & \text{\Large 0} & \\
& & & \boxed{2} & & & & \\
& & & & \boxed{\begin{matrix} 2 & 1 \\ & 2 \end{matrix}} & & & \\
& & & & & \boxed{\begin{matrix} 3 & 1 & \\ & 3 & 1 \\ & & 3 \end{matrix}} & & \\
& \text{\Large 0} & & & & & \boxed{\begin{matrix} 3 & 1 & \\ & 3 & 1 \\ & & 3 \end{matrix}} & \\
& & & & & & & \boxed{\begin{matrix} 3 & 1 & \\ & 3 & 1 \\ & & 3 \end{matrix}}
\end{pmatrix}$$

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Q Is $A = \begin{pmatrix} 2 & -2 & 14 \\ 0 & 3 & -7 \\ 0 & 0 & 2 \end{pmatrix}$ diagonalizable?

A $c_A(x) = \det(xI - A$