Lecture 48

Monday, April 27, 2015 10:04 AM

<u>Recall</u> Rational Canonical Form

linear transformation $T: V \to V$, V a fin dim F-vs. $\exists$ basis of V s.t. the matrix for T wrt that basis was a direct sum of companion matrices $C_{a_i(x)}$, where $V \cong \bigoplus F[x]/(a_i(x))$

& $a_1(x) \mid a_2(x) \mid \cdots \mid a_m(x)$ are monic poly's in $F[x]$.

If $a(x) = x^k + b_{k-1}x^{k-1} + \cdots + b_0$ then

$$C_{a(x)} = \begin{pmatrix} 0 & 0 & & 0 & -b_0 \\ 1 & 0 & & 0 & -b_1 \\ 0 & 1 & \cdots & 0 & -b_2 \\ 0 & 0 & & 0 & -b_3 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & 1 & -b_{k-1} \end{pmatrix}.$$

<u>Recall</u> $S, T: V \to V$ linear maps are <u>similar</u> if $\exists$ linear iso $U: V \to V$ s.t.

$$S = UTU^{-1}.$$

Thm TFAE: (1) S, T are similar

(2) $V_S \cong V_T$ as $F[x]$-modules

V_S: V w/ x acting as S

V_T: V w/ x acting as T

(3) S & T have the same RCF.

~~RCF~~

Pf (1) $\Rightarrow$ (2): Take $U \in GL(V)$ s.t. $S = UTU^{-1}$.

Then $U : V_T \longrightarrow V_S$ is an iso of $F[x]$-modules.

Indeed, $U(x \cdot v) = U(Tv) = (UT)(v)$

$= (SU)(v) = S(Uv) = x \cdot (Uv)$.

(2) $\Rightarrow$ (3): Same inv't factors $\Rightarrow$ same RCF.

(3) $\Rightarrow$ (1): $\exists$ bases s.t. S, T have same matrix so S, T similar. $\square$

Cor $A, B \in M_n(F)$, $F \subseteq K$, K another field.

A, B are similar over $K \Leftrightarrow$ similar over F. $\square$

Note RCF of matrix A always has entries in the smallest subfield containing entries of A.

RCF

characteristic poly   minimal polynomial

Lemma ① The char poly of a companion matrix $C_{a(x)}$ is $a(x)$.

② If $M = \begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_m \end{pmatrix}$ is a block sum of A_i, then $c_M(x) = \prod c_{A_i}(x)$

pf ② $xI - M = \bigoplus xI - A_i$ ✓

① $a(x) = x^k + b_{k-1}x^{k-1} + \cdots + b_0$

$$xI - C_{a(x)} = \begin{pmatrix} x & & & & b_0 \\ -1 & x & & & b_1 \\ & -1 & x & & b_2 \\ & & & \ddots & \vdots \\ & & & -1 & x+b_{k-1} \end{pmatrix}$$

Expand along final column:

$$(-1)^{k-1} b_0 \det \begin{pmatrix} -1 & x & & & \\ & -1 & x & & \\ & & -1 & \ddots & \\ & & & & x \\ & & & & -1 \end{pmatrix} = (-1)^{k-1} b_0 (-1)^{k-1} = b_0$$

$$(-1)^{k} b_1 \det \begin{pmatrix} x & & & & \\ & -1 & x & & \\ & & -1 & x & \\ & & & -1 & \\ & & & \ddots & x \\ & & & & -1 \end{pmatrix} = (-1)^{k} b_1 x (-1)^{k-2} = b_1 x$$

$$+ \quad \vdots$$

$$\text{char poly} = \det(xI - C_{a(x)}) = a(x)$$

$\square$

Thm If $a_1(x), \ldots, a_m(x)$ are the invt factors of the $F[x]$-mod V_T, $T: V \to V$, then

$$c_T(x) = \prod_{i=1}^{m} a_i(x).$$

Pf Let $B = \begin{pmatrix} C_{a_1(x)} & & \\ & \ddots & \\ & & C_{a_m(x)} \end{pmatrix}$ be the RCF of T.

By the lemma, $c_B(x) = \prod a_i(x)$.

But T & B are similar, thus have the same characteristic polynomial! $\blacksquare$

Cor · $m_T(x) \mid c_T(x)$

· $c_T(x) \mid m_T(x)^N \implies m_T(x), c_T(x)$ have same roots.

Algorithm for computing RCF : Reading

Sometimes tricks are enough:

$$A = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad xI - A = \begin{pmatrix} x & 1 & 1 \\ 0 & x & 0 \\ 1 & 0 & x \end{pmatrix}$$

$$c_A(x) = x \det \begin{pmatrix} x & 1 \\ 1 & x \end{pmatrix} = x(x^2-1) = x(x-1)(x+1).$$

$\Longrightarrow$ $m_A(x) = x(x-1)(x+1)$ b/c it divides $c_A(x)$ & has the same roots.

~~Thus the invariant factors~~

Since $\displaystyle\prod_{i=1}^{m} a_i(x) = \left(\prod_{i=1}^{m-1} a_i(x)\right) \cdot m_A(x) \underset{\text{Thm}}{=} c_A(x)$,

get $m_A(x)$ is the unique inv't factor.

We get $F_A^3 \cong F[x]/(x(x-1)(x+1)) = F[x]/(x^3-x)$

and RCF for A is $\boxed{\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}$

Finite abelian groups	torsion $F[x]$-modules
order n	char poly $c(x)$
$\mathbb{Z}/a_1\mathbb{Z} \times \cdots \times \mathbb{Z}/a_m\mathbb{Z}$	$\bigoplus F[x]/(a_i(x))$
a_i's invt factors, $n = \prod a_i$	
exponent a_m (generator of annihilator)	min poly $m_T(x)$
elementary divisors $\sim\sim\sim$	?