Lecture 47

Friday, April 24, 2015 10:01 AM

F a field

V a F-vector space

$T: V \longrightarrow V$ linear (Recall has an associated matrix for each basis of V)

$\Big\downarrow$

$F[x]$-module structure on V in which

$$x \cdot v = Tv$$

$$p(x) \cdot v = \big(p(T)\big)(v)$$

Idea Study T via the structure theorem for modules over a PID applied to the $F[x]$-module V.

Assume V is finite dimensional. Then V is a finitely generated torsion $F[x]$-module.

Thus
$$V \cong F[x]/(a_1(x)) \oplus F[x]/(a_2(x)) \oplus \cdots \oplus F[x]/(a_m(x))$$

where $a_i(x) \neq 0$, $a_i(x) \notin F[x]^\times = F^\times = F \setminus \{0\}$, and

$$a_1(x) \mid a_2(x) \mid \cdots \mid a_m(x).$$

Assume $a_i(x)$ are monic whence the inv^t factors $a_1, \ldots, a_m$ are unique.

Defn $\lambda \in F$ is an eigenvalue of T if $\exists v \in V$ s.t. $Tv = \lambda v$; then v is an eigenvector of T, $\{v \in V \mid Tv = \lambda v\}$ is the eigenspace of T corresponding to λ.

Prop TFAE: ① λ is an eigenvalue of T

② $\lambda I - T : V \to V$ is singular (kernel $\neq 0$)

③ $\det(\lambda I - T) = 0$

Pf $Tv = \lambda v \iff 0 = \lambda v - Tv = (\lambda I - T)(v)$

$\iff v \in \ker(\lambda I - T)$. $\square$

Defn The polynomial $c_T(x) = \det(xI - T)$ is the characteristic polynomial of T

Note · $\{\text{Roots of } c_T(x)\} = \{\text{eigenvalues of } T\}$

· $c_T(x)$ is monic of deg $n = \dim V$

$$\operatorname{Ann}(V) = \{ p(x) \in F[x] \mid p(x) \cdot V = 0 \}$$

$\uparrow$ $F[x]$-mod

$$= \operatorname{Ann}\left(F[x]/(a_1) \oplus \cdots \oplus F[x]/(a_m) \right)$$

$$= (a_m(x))$$

<u>Defn</u> The unique monic generator of $\mathrm{Ann}(V)$ in $F[x]$ is called the <u>minimal polynomial</u> of T, denoted

$m_T(x)$ = largest invariant factor of V.

(In particular, every invt factor divides $m_T(x)$.)

<u>Thm</u> $c_T(x) = \prod_{i=1}^{m} a_i(x)$.

<u>Cor</u> [Cayley-Hamilton Theorem] $m_T(x) \mid c_T(x)$.

<u>Cor</u> $c_T(x) \mid m_T(x)^N$ for some $N > 0$ so

$c_T(x)$ & $m_T(x)$ have the same roots

so $\{\text{Roots of } m_T(x)\} = \{\text{eigenvalues of } T\}$

Primary tool: <u>Rational Canonical Form</u>.

<u>Idea</u> Choose a basis for $F[x]/(a(x)) = F[x]/(x^k + b_{k-1}x^{k-1} + \dots + b_0)$

$1, \bar{x}, \bar{x}^2, \bar{x}^3, \dots, \bar{x}^{k-1}$

<u>Note</u>

$x \cdot 1 = \bar{x}$

$x \cdot \bar{x} = \bar{x}^2$

$x \cdot \bar{x}^2 = \bar{x}^3$

$\vdots$

$x \cdot \bar{x}^{k-2} = \bar{x}^{k-1}$

$x \cdot \bar{x}^{k-1} = \bar{x}^k$

$= -b_0 - b_1\bar{x} - b_2\bar{x}^2 - \dots - b_{k-1}\bar{x}^{k-1}$

Associated matrix for $x\cdot : F[x]/(a(x)) \longrightarrow F[x]/(a(x))$

wrt the basis $1, \bar{x}, \bar{x}^2, \dots, \bar{x}^{k-1}$:

$$\begin{pmatrix} 0 & 0 & 0 & & 0 & -b_0 \\ 1 & 0 & 0 & \cdots & 0 & -b_1 \\ 0 & 1 & 0 & & 0 & -b_2 \\ 0 & 0 & 1 & & 0 & -b_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & & 1 & -b_k \end{pmatrix} = C_{a(x)}$$

the companion matrix of $a(x)$.

If $V = \bigoplus_{i=1}^{m} F[x]/(a_i(x))$, then x acts via

$$\left(\begin{array}{c|c|c|c} C_{a_1(x)} & 0 & 0 & 0 \\ \hline 0 & C_{a_2(x)} & 0 & 0 \\ \hline 0 & 0 & \ddots & 0 \\ \hline 0 & 0 & 0 & C_{a_m(x)} \end{array}\right)$$

wrt the basis $\mathcal{B} = \mathcal{B}_1 \sqcup \mathcal{B}_2 \sqcup \dots \sqcup \mathcal{B}_m$ where $\mathcal{B}_i$'s are bases for $F[x]/(a_i(x))$.

$\uparrow$ i-th basis vectors in i-th coord, 0's in other coords.

<u>Defn</u> A matrix is in <u>rational</u> <u>canonical</u> <u>form</u> if it is the direct sum of companion matrices for $a_1(x), \ldots, a_m(x)$ monic of deg ≥ 1 w/

$$a_1 \mid a_2 \mid \cdots \mid a_m .$$

<u>Thm</u> If V is a fin. dim. F-vs, $T: V \to V$ linear, then ① $\exists$ basis for V wrt which the matrix for T is in rational canonical form

② The rat'l can form is unique.

<u>Pf</u> Just argued ①. ② follows from uniqueness in structure thm for modules / PID. $\square$