# Lecture 45

Tuesday, April 21, 2015 10:03 AM

## Modules over a PID

[Invariant factor form of the structure thm]

**Thm** $R$ a PID, $M$ finitely generated $R$-module.

Then $M \cong R^r \oplus R/(a_1) \oplus R/(a_2) \oplus \cdots \oplus R/(a_m)$

for some $r \geq 0$ & $a_i \neq 0$, $a_i \notin R^\times$ s.t.

$$a_1 \mid a_2 \mid \cdots \mid a_m$$

free rank

invariant factors

and $r$, $a_i$ are unique.

**Cor** $R = \mathbb{Z}$: every f.g. abelian gp is $\cong$ to

some $\mathbb{Z}^r \times \mathbb{Z}/a_1\mathbb{Z} \times \cdots \times \mathbb{Z}/a_m\mathbb{Z}$

w/ $r \geq 0$, $a_1 \mid a_2 \mid \cdots \mid a_m$.

**Def'n** A left $R$-module $M$ is Noetherian if $\forall$ chain of submodules $M_1 \leq M_2 \leq M_3 \leq \cdots \leq M$, $M_k = M_m$ for $k \geq m$. The ring $R$ is Noetherian if it is Noetherian as an $R$-module.

<u>Thm</u> $M$ a left $R$-module. TFAE:

① $M$ is Noetherian.

② Every nonempty set of submodules of $M$ contains a max'l elt.

③ Every submodule of $M$ is finitely generated.

<u>Pf</u> ① $\Rightarrow$ ② : $\Sigma$ a nonempty set of submods of $M$.

Take $M_1 \in \Sigma$. If $M_1$ is max'l, we're done. If not, $\exists\, M_2 \in \Sigma$ w/ $M_1 \subsetneq M_2$. If $M_2$ is max'l, we're done. If not, $\exists M_3 \in \Sigma$ w/ $M_1 \subsetneq M_2 \subsetneq M_3$, etc.

If this does not terminate, we get an infinite ascending chain of submods ☠.

② $\Rightarrow$ ③ : Take $N \leq M$, $\Sigma = \{\text{f.g. submods of } N\}$.

$0 \in \Sigma \neq \emptyset$. $\Rightarrow \Sigma$ contains max'l $N'$.

If $N' \neq N$, then $\exists x \in N \setminus N'$ & $N' + Rx \in \Sigma$, $N' \subsetneq N' + Rx$ ☠. Thus $N = N'$ is f.g.

③ $\Rightarrow$ ① : $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots \subseteq M$.

Let $N = \bigcup M_i \subseteq M$ so $N$ is generated by finite set $x_1, \ldots, x_n$. Each $x_i \in M_{j_i}$

Let $m = \max\{j_1, j_2, \ldots, j_n\}$. Then $x_i \in M_m$

$\Rightarrow N = M_m = M_k$ for $k \geq m$. $\square$

<u>Cor</u> $R$ a PID, then every nonempty set of ideals in $R$ has a max'l elt.

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<u>Prop</u> $R$ an integral domain. $M$ a free $R$-module of rank $n < \infty$. Then any $n+1$ elts in $M$ are $R$-linearly dependent: i.e. $\forall y_1, \ldots, y_{n+1} \in M$, $\exists\, r_1, \ldots, r_{n+1} \in M$ not all $0$ s.t. $\sum r_i y_i = 0$.

<u>Pf</u> $M \cong R^n \subseteq (\mathrm{Frac}(R))^n$

Thus $\exists\, q_i \in \mathrm{Frac}(R)$ not all $0$ s.t. $\sum q_i y_i = 0$

Clear denominators to get an $R$-linear dependence. $\square$

**Defn** $R$ an integral domain $\mathrm{Tor}(M) = \{ x \in M \mid rx = 0 \text{ for some } r \in R \setminus 0 \} \leq M$

is the torsion submodule of $M$.

**Main Lemma** $R$ a PID, $M$ a free $R$-module of finite rank $n$, $N \leq M$. Then

1. $N$ is free of rank

<u>Defn</u> $R$ an int dom. The <u>rank</u> of an $R$-mod $M$ is the maximum number of $R$-lin ind elts of $M$.

<u>Thm</u> $R$ a PID, $M$ a free $R$-mod of finite rank $n$, $N \leq M$. Then

① $N$ is free of rk $m \leq n$

② $\exists$ basis $y_1, \ldots, y_n$ of $M$ so that $a_1 y_1, \ldots, a_m y_m$ is a basis of $N$ for $a_i \in R \setminus 0$ & $a_1 | a_2 | \cdots | a_m$.

<u>Pf</u> If $N = 0$, the thm is true. Assume $N \neq 0$. Let

$$\Sigma = \{ \varphi(N) \mid \varphi \in \mathrm{Hom}_R(M, R) \}$$
$$= \{ (a_\varphi) \mid \varphi \in \text{—''—} \} \text{ b/c } R \text{ is a PID.}$$

$$\ni (0) \text{ so nonempty, so } \Sigma \text{ has a max'l elt } (a_\nu) = (a_1)$$

Take $y \in N$ s.t. $\nu(y) = a_1$.

$a_1 \neq 0$ b/c $\Sigma \setminus \{0\} \neq \emptyset$ b/c some $\pi_i N \neq 0$ for $\pi_i$ proj'n onto $i$<u>th</u> coord for basis $x_1, \ldots, x_n$ of $M$.

<u>Claim</u> $a_1 | \varphi(y)$ $\forall \varphi \in \mathrm{Hom}(M, R)$. Take $d$ s.t. $(d) = (a_1, \varphi(y))$.

$d | a_1, \varphi(y)$, $d = r_1 a_1 + r_2 \varphi(y)$. Take $\psi = r_1 \nu + r_2 \varphi : M \to R$.

$\psi(y) = r_1 a_1 + r_2 \varphi(y) = d \Rightarrow d \in \psi(N)$ so $(d) \subseteq \psi(N)$.

Since $d | a_1$, $(a_1) \subseteq (d) \subseteq \psi(N)$. By maximality of $(a_1)$,

$\underbrace{(a_1) = (d)}_{} = \psi(N)$.

$\Rightarrow a_1 | \varphi(y)$.

In particular, $a_1 \mid \pi_i(y)$ $\forall i$, $\pi_i(y) = a_1 b_i$. Define

$$y_1 = \sum_{i=1}^{n} b_i x_i$$

Then $a_1 y_1 = y$. $a_1 = \nu(y) = \nu(a_1 y_1) = a_1 \nu(y_1) \Rightarrow \nu(y_1) = 1$.

<u>Claim</u> (a) $M = Ry_1 \oplus \ker \nu$

(b) $N = Ra_1 y_1 \oplus (N \cap \ker \nu)$

(Thus $y_1$ a basis elt for $M$, $a_1 y_1$ basis elt for $N$.)

(a): $x \in M$. $x = \nu(x) y_1 + (x - \nu(x) y_1)$

$$\nu(x - \nu(x) y_1) = \nu(x) - \nu(x)\nu(y_1)$$
$$= \nu(x) - \nu(x) \cdot 1$$
$$= 0$$

so $x - \nu(x) y_1 \in \ker \nu \Rightarrow x \in Ry_1 + \ker \nu$

$\Rightarrow M = Ry_1 + \ker \nu$

In order for the sum to be direct, we need $Ry_1 \cap \ker \nu = 0$.

Suppose $ry_1 \in \ker \nu \Rightarrow 0 = \nu(ry_1) = r\nu(y_1) = r \Rightarrow ry_1 = 0$ ✓

Thus $M = R \oplus \ker \nu$.

(b): Since $(a_1) = \nu(N)$, $a_1 \mid \nu(x')$ $\forall x' \in N$.

If $\nu(x') = ba_1$ then $x' = \nu(x') y_1 + (x' - \nu(x') y_1)$

$$= ba_1 y_1 + \underbrace{(x' - ba_1 y_1)}_{\in \ker \nu \cap N}$$

$\Rightarrow N = Ra_1 y_1 + (N \cap \ker \nu)$

Direct? Same as above.