# Lecture 44

Monday, April 20, 2015 10:01 AM

__Recall__ $M$ a right $R$-mod, $N$ a left $R$-mod,

we constructed $M \otimes_R N$ as an abelian group.

If $M$ is an $(S,R)$-bimodule, then $M \otimes_R N$ has a natural left $S$-mod str.

If $R$ is commutative, $M, N$ are $(R,R)$-bimodules

so $M \otimes_R N$ is an $R$-module.

Universal "source" of $R$-bilinear forms

$$\begin{tikzcd} M \times N \arrow[r] \arrow[dr, "\langle\, ,\, \rangle"', "R\text{-bilin}"'] & M \otimes_R N \arrow[d, dashed, "\exists!\ R\text{-mod}"] \\ & L \end{tikzcd}$$

__e.g.__ $M \otimes_R R \cong M \cong R \otimes_R M$

$m \otimes r = mr \otimes 1 \longmapsto mr$

Only OK to define homs on simple tensor if the assoc map $(m,n) \longmapsto ?$ is a $R$-balanced / $R$-bilinear map.

e.g. Extension of scalars:

$S \subseteq R$ subring of $R$

$M$ a left $S$-module

$R$ has a canonical $(R,S)$-bimodule str.

$R \otimes_S M$ is called the extn of scalars of $M$ from $S$ to $R$.

Considering $R \otimes_S M$ as an $S$-module, we see $M$ an $S$-submodule via

$$M \subseteq R \otimes_S M$$

$$m \longmapsto 1 \otimes m$$

$$sm \longmapsto 1 \otimes sm = s \otimes m = s \cdot (1 \otimes m)$$

In fact, $R \otimes_S \bigoplus_A S \cong \bigoplus_A R$.

e.g. $V \cong \mathbb{R}^n$ an $\mathbb{R}$-vector space, then

$$\mathbb{C} \otimes_{\mathbb{R}} V \cong \mathbb{C}^n.$$

<u>Prop</u> $S \subseteq R$ subring, $M_a$ $S$-modules for $a \in A$.

Thm $R \underset{S}{\otimes} \bigoplus_A M_a \cong \bigoplus_A (R \underset{S}{\otimes} M_a)$.

<u>Cor</u> $R \underset{S}{\otimes} \bigoplus_A S \cong \bigoplus_A R$

<u>Pf</u> $R \underset{S}{\otimes} \bigoplus_A S = \bigoplus_A (R \underset{S}{\otimes} S) \cong \bigoplus_A R$. $\square$

since $S$ is a "unit" for $\underset{S}{\otimes}$.

<u>Pf Prop</u> Define $R \underset{S}{\otimes} M_a \xrightarrow{\iota_a} R \underset{S}{\otimes} \bigoplus_A M_a$

$r \otimes m \longmapsto r \otimes$ (A-tuple of 0's except $m$ in the $a$-th pos'n)

Given $\{f_a : R \underset{S}{\otimes} M_a \longrightarrow N \mid a \in A\}$

$\exists! \; f : R \underset{S}{\otimes} \bigoplus_A M_a \longrightarrow N$ s.t. $f \circ \iota_a = f_a$

$r \otimes (m_a) \longmapsto \sum f_a(r \otimes m_a)$

By univ prop, $R \underset{S}{\otimes} \bigoplus_A M_a \cong \bigoplus_A (R \otimes M_a)$. $\square$

# Associativity & Multilinearity

For simplicity, assume $R$ comm, $M, N, L$ $R$-modules.

Thm $(M \otimes_R N) \otimes_R L \cong M \otimes_R (N \otimes_R L)$.

Pf Fix $l \in L$. For $m \in M$, $n \in N$

$$((m,n), l) \longmapsto m \otimes (n \otimes l)$$

is bilinear in $m, n$:

$$(m+m') \otimes (n \otimes l) = m \otimes (n \otimes l) + m' \otimes (n \otimes l)$$

$$m \otimes ((n+n') \otimes l) = m \otimes (n \otimes l + n' \otimes l)$$

$$= m \otimes (n \otimes l) + m \otimes (n' \otimes l)$$

$$mr \otimes (n \otimes l) = m \otimes (r(n \otimes l)) = m \otimes ((rn) \otimes l)$$

So for each $l \in L$ we get $M \otimes_R N \xrightarrow{\varphi_l} M \otimes_R (N \otimes_R L)$

$$(M \otimes_R N) \times L \longrightarrow M \otimes_R (N \otimes_R L)$$

$$(m \otimes n, l) \longmapsto \varphi_l(m \otimes n) = m \otimes (n \otimes l)$$

Check bilinear

Thus, we have a hom $(M \otimes_R N) \otimes_R L \longrightarrow M \otimes_R (N \otimes_R L)$.

In the same fashion, we get

$$M \otimes (N \otimes L) \longrightarrow (M \otimes N) \otimes L$$
$$m \otimes (n \otimes l) \longmapsto (m \otimes n) \otimes l$$

which is manifestly a 2-sided inverse of the previous hom.

$\square$

Thus $M_1 \otimes_R M_2 \otimes_R \cdots \otimes_R M_n$ makes sense for $n \geq 0$.

[Note $\bigotimes_{\emptyset} = R$]

Defn $M_1, \ldots, M_n$ are $R$-modules, $R$ a comm ring.
$M_1 \times \cdots \times M_n \xrightarrow{\varphi} L$ is $R$-multilinear if
it is additive in each coordinate and

$$\varphi(m_1, \ldots, m_{i-1}, rm_i, m_{i+1}, \ldots, m_n) = r\varphi(m_1, \ldots, m_n)$$

Cor $\left\{ \begin{matrix} R\text{-multilinear} \\ M_1 \times \cdots \times M_n \longrightarrow L \end{matrix} \right\} \xleftrightarrow{\cong} \left\{ \begin{matrix} R\text{-mod homs} \\ M_1 \otimes_R \cdots \otimes_R M_n \longrightarrow L \end{matrix} \right\}.$

<u>Thm</u> [ $\otimes$-Hom adjunction ] $R$ commutative, $M, N, L$ $R$-modules. Then

$$\mathrm{Hom}_R(M \otimes_R N, L) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, L)).$$

<u>Pf</u> Suppose that $\langle , \rangle : M \times N \longrightarrow L$ is $R$-bilinear.

Define
$$\begin{aligned} M &\longrightarrow \mathrm{Hom}_R(N, L) \\ m &\longmapsto (n \longmapsto \langle m, n \rangle) \end{aligned}$$

<u>Check</u> This is an $R$-mod hom!

Given $R$-mod hom $\varphi : M \longrightarrow \mathrm{Hom}_R(N, L)$

define
$$\begin{aligned} \langle , \rangle_\varphi : M \times N &\longrightarrow L \\ (m, n) &\longmapsto (\varphi(m))(n) \end{aligned}$$

<u>Check</u> $\langle , \rangle_\varphi$ is bilinear.

We have inverse maps

$$\begin{tikzcd} \mathrm{Bilin}(M \times N, L) \arrow[r, leftrightarrow, "\cong"] \arrow[d, leftrightarrow, "\cong"'] & \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, L)) \\ \mathrm{Hom}_R(M \otimes_R N, L) & \end{tikzcd}$$

<u>Check</u> All assignments are $R$-mod homs.