

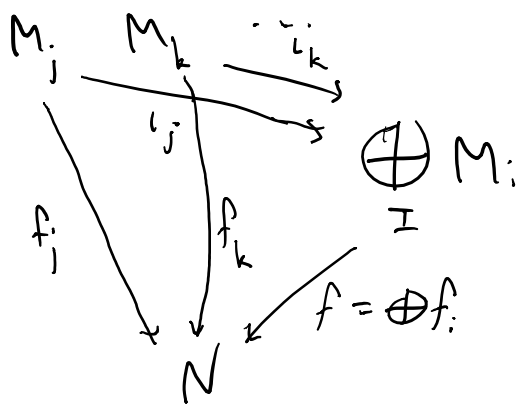
Lecture 43

Friday, April 17, 2015 10:02 AM

Pf of Hom / \oplus Thm

Claim $\text{Hom}_{\mathbb{R}}(\bigoplus_I M_i, N) \cong \prod_I \text{Hom}_{\mathbb{R}}(M_i, N)$

Given a \mathbb{R} -mod hom $\bigoplus_I M_i \xrightarrow{f} N$, we have



$$f_i = f \circ i_j$$

Define $\text{Hom}_{\mathbb{R}}(\bigoplus_I M_i, N) \longrightarrow \prod_I \text{Hom}_{\mathbb{R}}(M_i, N)$
 $f \longmapsto (f_i)_{i \in I}$

check: this is a map of ab gps (\mathbb{R} -mods if \mathbb{R} comm)

$(f_i)_{i \in I} \longmapsto \bigoplus f_i$ is a 2-sided inverse
 to $f \longmapsto (f_i)$.

$f_i : M_i \rightarrow N$ \mathbb{R} -mod hom



Tensor products R ring w/ 1.

M a right R -module, N a left R -module

$$\begin{array}{l} M \times R \longrightarrow M \\ (m, r) \longmapsto mr \end{array} \qquad \begin{array}{l} R \times N \longrightarrow N \\ (r, n) \longmapsto rn \end{array}$$

An R -balanced form on $M \times N$ is a function

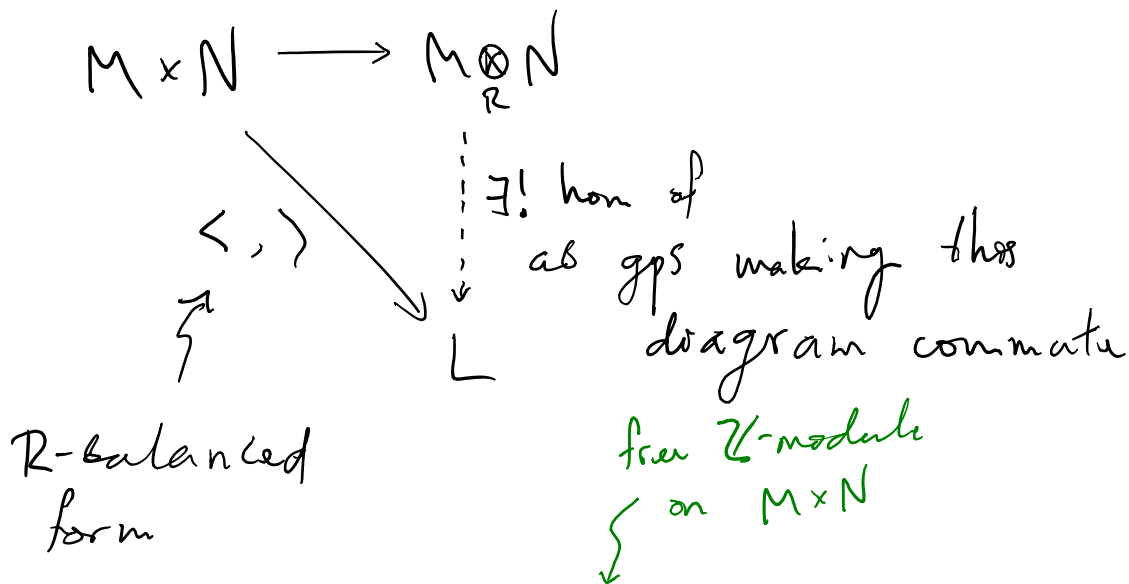
$$\begin{array}{l} M \times N \longrightarrow L \text{ to an abelian group } L \\ (m, n) \longmapsto \langle m, n \rangle \end{array}$$

$$\begin{array}{l} \text{s.t. } \langle m+m', n \rangle = \langle m, n \rangle + \langle m', n \rangle \\ \langle m, n+n' \rangle = \langle m, n \rangle + \langle m, n' \rangle \\ \langle mr, n \rangle = \langle m, rn \rangle \end{array} \left. \begin{array}{l} \\ \\ \end{array} \right\} \begin{array}{l} \text{additivity} \\ \\ R\text{-balanced} \end{array}$$

Goal Construct an object $M \otimes_R N$ s.t.

$$\left\{ \begin{array}{l} R\text{-balanced forms } M \times N \longrightarrow L \\ \cong \left\{ \text{homs } M \otimes_R N \longrightarrow L \right\} \end{array} \right\}$$

We get such a correspondence if $M \otimes_R N$ satisfies the following univ prop:



Defn $M \otimes_R N := \mathbb{Z} \cdot (M \times N) / \mathcal{R}$

\mathcal{R} is a subgroup of "relations"

$$\mathcal{R} = \left\langle \begin{array}{l} (m+m', n) - (m, n) - (m', n) \\ (m, n+n') - (m, n) - (m, n') \\ (mr, n) - (m, rn) \end{array} \right. \left. \begin{array}{l} m, m' \in M \\ n, n' \in N \\ r \in R \end{array} \right\rangle$$

Thm $M \otimes_R N$ satisfies the univ prop.

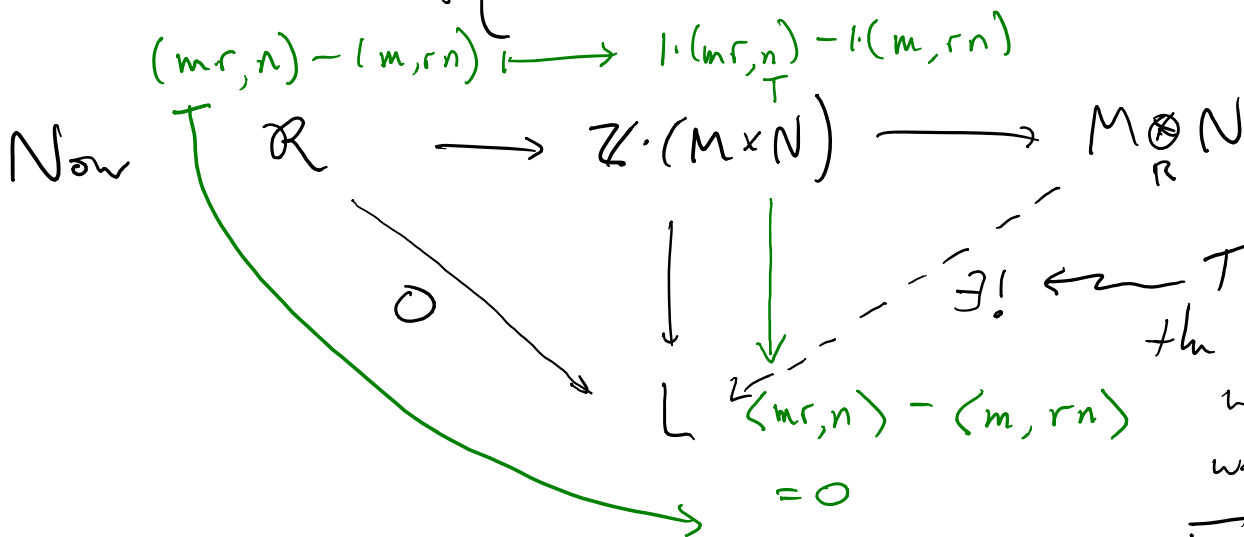
Pf Let $\underbrace{m \otimes n}_{\text{simple tensor}}$ be the coset of $1 \cdot (m, n)$ in $M \otimes_{\mathbb{R}} N$.

Define $M \times N \longrightarrow M \otimes_{\mathbb{R}} N$ the canonical reduction map restricted to $1 \cdot (M \times N)$.
 $(m, n) \longmapsto m \otimes n$

Given an \mathbb{R} -balanced form $\langle , \rangle : M \times N \longrightarrow L$
 the univ prop of the free \mathbb{Z} -module $\mathbb{Z} \cdot (M \times N)$
 gives a \mathbb{Z} -mod hom $\mathbb{Z} \cdot (M \times N) \longrightarrow L$
 $1 \cdot (m, n) \longmapsto \langle m, n \rangle$

$$\begin{aligned} (m, n) &\longmapsto 1 \cdot (m, n) \\ M \times N &\longrightarrow \mathbb{Z} \cdot (M \times N) \end{aligned}$$

\langle , \rangle \searrow $\exists!$ \mathbb{Z} -mod hom



$\exists!$ \longleftarrow This is the map we wanted!



Does $M \otimes_R N$ have a module structure?

Def'n Rings S, R w/1 define an (S, R) -bimodule

M to be a left S -module, right R -module s.t.

$$s(mr) = (sm)r \quad \text{for all } s \in S, r \in R, m \in M,$$

e.g. ① If R is commutative, a left R -module M inherits an (R, R) -bimodule structure in which $mr = rm$.

Note $(mr)s = (rm)s = s(rm) = (sr)m$
 $m(rs) = (rs)m$ need R to be commutative!

② Suppose $f: R \rightarrow S$ ring hom.

S is an (S, R) -bimod via

$$s \cdot x = sx, \quad x \cdot r = xf(r)$$

$$s \in S, x \in S, r \in R$$

③ $I \trianglelefteq R$, R/I is an $(R/I, R)$ -bimod

via $f: R \rightarrow R/I$ reduction mod I .

Prop If M is an (S, R) -bimod, and N is a left R -mod, then $M \otimes_R N$ is an S -module

via $s \left(\sum_{\text{finite}} m_i \otimes n_i \right) = \sum (sm_i) \otimes n_i$ \square

Special case R comm, M, N R -mods have canonical (R, R) -bimod str. In this case,

$M \otimes_R N$ is an R -module.

$m \in M, n \in N, r \in R$

$mr \otimes n = m \otimes rn$

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$(rm) \otimes n$

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$r(m \otimes n)$

Def'n R comm ring w/1. An R -bilinear form

$\langle , \rangle : M \times N \rightarrow L$, L an R -module satisfies additivity relations & $\langle rm, n \rangle = r \langle m, n \rangle = \langle m, rn \rangle$

We get a univ property that for every
 R -bilin form $\langle, \rangle : M \times N \rightarrow L \exists! R\text{-mod}$
 hom $M \otimes_R N \rightarrow L$ s.t.

