Lecture 43

Friday, April 17, 2015 10:02 AM

Pf of Hom / $\bigoplus$ Thm

Claim $\mathrm{Hom}_R\left(\bigoplus_I M_i, N\right) \cong \prod_I \mathrm{Hom}_R(M_i, N)$

Given a R-mod hom $\bigoplus_I M_i \xrightarrow{f} N$, we have $f_j = f \circ \iota_j$

$$\begin{tikzcd}
M_j \arrow[rr, "\iota_j"] \arrow[dr, "f_j"'] & M_k \arrow[r, "\iota_k"] \arrow[d, "f_k"] & \cdots \; \bigoplus_I M_i \arrow[dl, "f = \bigoplus f_i"] \\
& N &
\end{tikzcd}$$

Define
$$\mathrm{Hom}_R\left(\bigoplus_I M_i, N\right) \longrightarrow \prod_I \mathrm{Hom}_R(M_i, N)$$
$$f \longmapsto (f_i)_{i \in I}$$

Check: this is a map of ab gps (R-mods if R comm)

$(f_i)_{i\in I} \longmapsto \bigoplus f_i$ is a 2-sided inverse to $f \longmapsto (f_i)$.

$f_i : M_i \longrightarrow N$ R-mod hom

$\blacksquare$

<u>Tensor products</u> R ring w/ 1.

M a right R-module, N a left R-module

$$M \times R \longrightarrow M \qquad R \times N \longrightarrow N$$
$$(m, r) \longmapsto mr \qquad (r, n) \longmapsto rn$$

An <u>R-balanced form</u> on $M \times N$ is a function

$M \times N \longrightarrow L$ to an abelian group L

$(m, n) \longmapsto \langle m, n \rangle$

s.t.
$$\left.\begin{array}{l} \langle m+m', n \rangle = \langle m, n \rangle + \langle m', n \rangle \\ \langle m, n+n' \rangle = \langle m, n \rangle + \langle m, n' \rangle \end{array}\right\} \text{additivity}$$
$$\left.\langle mr, n \rangle = \langle m, rn \rangle \right\} R\text{-balanced}$$

<u>Goal</u> Construct an object $M \underset{R}{\otimes} N$ s.t.

$$\{R\text{-balanced forms } M \times N \to L\}$$
$$\underset{\cong}{\longleftrightarrow} \{\text{homs } M \underset{R}{\otimes} N \longrightarrow L\}$$

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We get such a correspondence if $M \otimes_R N$ satisfies the following univ prop:

$$\begin{tikzcd} M \times N \arrow[r] \arrow[dr, "{\langle\, ,\, \rangle}"'] & M \otimes_R N \arrow[d, dashed] \\ & L \end{tikzcd}$$

$\exists!$ hom of ab gps making this diagram commute

R-balanced form

free $\mathbb{Z}$-module on $M \times N$

<u>Defn</u> $M \otimes_R N := \mathbb{Z} \cdot (M \times N) / \mathcal{R}$

subgp of "relations"

$$\mathcal{R} = \left\langle \begin{array}{c} (m+m', n) - (m,n) - (m',n) \\ (m, n+n') - (m,n) - (m,n') \\ (mr, n) - (m, rn) \end{array} \middle| \begin{array}{c} m, m' \in M \\ n, n' \in N \\ r \in R \end{array} \right\rangle$$

<u>Thm</u> $M \otimes_R N$ satisfies the univ prop.

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Pf Let $\underbrace{m\otimes n}_{\text{simple tensor}}$ be the coset of $1\cdot(m,n)$ in $M\otimes_R N$.

Define

$$\begin{array}{ccc} M\times N & \longrightarrow & M\otimes_R N \\ (m,n) & \longmapsto & m\otimes n \end{array}$$

the canonical reduction map restricted to $1\cdot(M\times N)$.

Given an R-balanced form $\langle\, , \rangle : M\times N \longrightarrow L$ the univ prop of the free $\mathbb{Z}$-module $\mathbb{Z}\cdot(M\times N)$ gives a $\mathbb{Z}$-mod hom

$$\begin{array}{ccc} \mathbb{Z}\cdot(M\times N) & \longrightarrow & L \\ 1\cdot(m,n) & \longmapsto & \langle m,n\rangle \end{array}$$

$$\begin{tikzcd}
M\times N \arrow[r] \arrow[dr, "{\langle\, , \rangle}"'] & \mathbb{Z}\cdot(M\times N) \arrow[d, dashed, "\exists!\ \mathbb{Z}\text{-mod hom}"] \\
& L
\end{tikzcd}$$

$(m,n)\longmapsto 1\cdot(m,n)$

$(mr,n)-(m,rn) \longmapsto 1\cdot(mr,n)-1\cdot(m,rn)$

Now

$$\begin{tikzcd}
\mathcal{R} \arrow[r] \arrow[drr, "0"'] & \mathbb{Z}\cdot(M\times N) \arrow[r] \arrow[dr] & M\otimes_R N \arrow[d, dashed, "\exists!"] \\
& & L
\end{tikzcd}$$

$\langle mr,n\rangle - \langle m,rn\rangle = 0$

$\exists!$ ← This is the map we wanted!

$\square$

Does $M \otimes_R N$ have a module structure?

Def'n Rings S, R w/1 define an (S,R)-bimodule M to be a left S-module, right R-module s.t.

$$s(mr) = (sm)r \quad \text{for all } s \in S,\ r \in R,\ m \in M.$$

e.g. (1) If R is commutative, a left R-module M inherits an (R,R)-bimodule structure in which $mr = rm$.

Note $(mr)s = (rm)s = s(rm) = (sr)m$

$= m(rs) = (rs)m$ need R to be commutative!

(2) Suppose $f: R \to S$ ring hom.

S is an (S,R)-bimod via

$$s \cdot x = sx, \quad x \cdot r = x f(r)$$

$$s \in S,\ x \in S,\ r \in R$$

(3) $I \trianglelefteq R$, R/I is an $(R/I, R)$-bimod via $f: R \to R/I$ reduction mod I.

<u>Prop</u> If M is an (S,R)-bimod, and N is a left R-mod, then $M \otimes_R N$ is an S-module

via $$s\left(\sum_{\text{finite}} m_i \otimes n_i\right) = \sum (sm_i) \otimes n_i \qquad \square$$

<u>Special case</u> R comm, M, N R-mods have canonical (R,R)-bimod str. In this case, $M \otimes_R N$ is an R-module.

$m \in M$, $n \in N$, $r \in R$

$$mr \otimes n = m \otimes rn$$
$$\| $$
$$(rm) \otimes n$$
$$\|$$
$$r(m \otimes n)$$

<u>Def'n</u> R comm ring w/ 1. An R-<u>bilinear</u> <u>form</u> $\langle\ ,\ \rangle : M \times N \longrightarrow L$, L an R-module satisfies additivity relations & $\langle rm, n\rangle = r\langle m,n\rangle = \langle m, rn\rangle$

We get a univ property that for every R-bilin form $\langle , \rangle : M \times N \to L$ $\exists!$ R-mod hom $M \otimes_R N \to L$ s.t.

$$\begin{tikzcd} M \times N \arrow[r] \arrow[dr] & M \otimes_R N \arrow[d, dashed, "\exists!"] \\ & L \end{tikzcd}$$

commutes.