Lecture 42

Wednesday, April 15, 2015 10:01 AM

$R$ a ring w/ 1.

$M$ an $R$-module, write $N \leq M$ if $N$ is a submodule

① $N_1, \dots, N_n \leq M$,

$$N_1 + N_2 + \cdots + N_n = \{a_1 + \cdots + a_n \mid a_i \in N_i\}$$

is the submodule of $M$ generated by $N_1, \dots, N_n$.

② $A \subseteq M$, define $RA = \{ r_1 a_1 + \cdots + r_n a_n \mid r_i \in R, \ a_i \in A, \ n \geq 0 \}$

If $A = \{a_1, \dots, a_k\}$ is finite, then

$$RA = Ra_1 + Ra_2 + \cdots + Ra_k$$

$RA$ is the submodule of $M$ generated by $A$.

i.e. spanned by $A$.

③ $N \leq M$ is <u>finitely generated</u> if $\exists$ finite $A \subseteq M$ s.t.

$N = RA$.

(4) $N \leq M$ is <u>cyclic</u> if $N = Ra$ for some $a \in M$.

<u>e.g.</u> $M = R$. Submodules of $R$ are ideals of $R$.

- finitely generated submodules of $R$
  = finitely generated ideals of $R$

- cyclic submodules of $R$
  = principal ideals of $R$.

(!) submodules of f.g. modules need not be f.g.

<u>e.g.</u> $M = R = \mathbb{Z}[x_1, x_2, \ldots]$.

$(x_1, x_2, x_3, \ldots) \leq M$

↑ not finitely generated!

<u>e.g.</u> $M = R^n$ is finitely generated

$e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$

↑ i-th coordinate

$M = R\{e_1, \ldots, e_n\}$

# Direct products, direct sums

$M, N, P$ R-modules

$$\begin{tikzcd}
 & P \arrow[dl, bend right, "f"'] \arrow[d, dashed, "\exists! f\times g"] \arrow[dr, bend left, "g"] & \\
M & M\times N \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & N
\end{tikzcd}$$

$M\times N$ is the product of $M, N$ in R-modules.

$$(f\times g)(p) = (f(p), g(p))$$

Categories: Prefix "co" means flip all the arrows.

$\underbrace{\text{"Coproduct"}}_{\text{direct sum}}$ ??

$$\begin{tikzcd}
M \arrow[r, "\iota_1"] \arrow[dr, bend right, "f"'] & M\oplus N \arrow[d, dashed, "\exists! f\oplus g"] & N \arrow[l, "\iota_2"'] \arrow[dl, bend left, "g"] \\
 & P &
\end{tikzcd}$$

$\iota$ = iota

\iota

$$M\oplus N = M\times N$$

$\iota_1 : M \longrightarrow M\oplus N$ $\quad$ $\iota_2 : N \longrightarrow M\oplus N$ $\quad$ $f\oplus g : M\oplus N \longrightarrow P$

$m \longmapsto (m, 0)$ $\quad$ $n \longmapsto (0, n)$ $\quad$ $(m,n) \longmapsto f(m) + g(n)$

By induction, $M_1 \times \cdots \times M_n \cong M_1 \oplus \cdots \oplus M_n$.

$\underline{Q}$ What are products / coproducts indexed by an arbitrary set?

$$\begin{tikzcd}
& P \arrow[dl, "f_i"'] \arrow[d, "f_j"] \arrow[dr, "f_k"] \arrow[rr, dashed, bend left, "\exists!"] & & \prod\limits_{i\in I} M_i \arrow[dlll, "\pi_i"] \arrow[dll, "\pi_j"] \arrow[dl, "\pi_k"] \arrow[d, "\cdots"] \\
M_i & M_j & M_k & \underbrace{\cdots}_{\text{other indices in an idex set } I}
\end{tikzcd}$$

The $\underline{\text{product}}$ of $M_i$, $i \in I$, is a module $\prod\limits_I M_i$ equipped w/ "projection" homs $\prod\limits_I M_i \xrightarrow[\pi_j]{} M_j$ s.t. $\forall$ R-mod $P$ and collection of homs $f_i : P \longrightarrow M_i$, $i \in I$,

$$\exists! \prod_I f_i : P \longrightarrow \prod_I f_i \quad \text{s.t.} \quad f_j = \pi_j \circ \prod_I f_i$$

$\forall j \in I$.

$$
\begin{tikzcd}
 & P \arrow[dl, "f_i"'] \arrow[d, "f_j"] \arrow[dr, "f_k"] \arrow[rr, dashed, "\exists!"] & & \prod_{i\in I} M_i \arrow[lll, bend left, "\pi_i"] \arrow[ll, bend left, "\pi_j"] \arrow[l, "\pi_k"] \\
M_i & M_j & M_k & \cdots
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
M_i \arrow[dr, "f_i"'] \arrow[rrr, bend left, "\iota_i"] & M_j \arrow[d, "f_j"] \arrow[rr, bend left, "\iota_j"] & M_k \arrow[dl, "f_k"] \arrow[r, "\iota_k"] & \bigoplus_I M_i \arrow[dll, dashed, "\exists!\ \bigoplus f_i"] \\
 & P & &
\end{tikzcd}
$$

$$\prod_I M_i = \{ (m_i)_{i\in I} \mid m_i \in M_i \}$$

$$\pi_j : \prod_I M_i \longrightarrow M_j \qquad\qquad \prod_I f_i : P \longrightarrow \prod_I M_i$$

$$(m_i) \longmapsto m_j \qquad\qquad p \longmapsto (f_i(p))_{i\in I}$$

$$\bigoplus_I M_i = \{ (m_i)_{i\in I} \mid m_i \in M_i,\ m_i = 0 \text{ for all but finitely many } i\in I \}$$

$$\iota_j : M_j \longrightarrow \bigoplus_I M_i \qquad\qquad \bigoplus f_i : \bigoplus_I M_i \longrightarrow P$$

$$m_j \longmapsto (m_i) \text{ where } m_i = \begin{cases} m_j & i = j \\ 0 & \text{o/w} \end{cases} \qquad\qquad (m_i) \longmapsto \sum_{i\in I} f_i(m_i)$$

well-defined b/c almost every $m_i = 0$.

<u>Def'n</u> An $R$-module $M$ is <u>free</u> on $A \subseteq M$ if $\forall x \in M$
$\exists!\ r_a \in R$, $a \in A$ s.t. $x = \sum_{a \in A} r_a \cdot a$

Call $A$ a <u>basis</u> of $M$.

When $M$ is free on $A$, $M \cong \bigoplus_A R = R^A$

$$\begin{tikzcd}
a \arrow[r, mapsto] & e_a & \\
A \arrow[r] \arrow[dr, "\varphi"'] & \bigoplus_A R \arrow[d, dashed, "\exists!"] & (r_a) \arrow[d, mapsto] \\
& N & \sum r_a \varphi(a)
\end{tikzcd}$$

<u>Thm</u>

- $\mathrm{Hom}_R\left(\bigoplus_I M_i, N\right) \cong \prod_I \mathrm{Hom}_R(M_i, N)$
- $\mathrm{Hom}_R\left(M, \prod_I N_i\right) \cong \prod_I \mathrm{Hom}_R(M, N_i)$

as $R$-modules / abelian gps.