

# Lecture 42

Wednesday, April 15, 2015 10:01 AM

$R$  a ring w/ 1.

$M$  an  $R$ -module, write  $N \leq M$  if  $N$  is a submodule

①  $N_1, \dots, N_n \leq M$ ,

$$N_1 + N_2 + \dots + N_n = \{ a_1 + \dots + a_n \mid a_i \in N_i \}$$

is the submodule of  $M$  generated by  $N_1, \dots, N_n$ .

②  $A \leq M$ , define  $RA = \{ r_1 a_1 + \dots + r_n a_n \mid r_i \in R, \left. \begin{array}{l} a_i \in A, \\ n \geq 0 \end{array} \right\}$

If  $A = \{ a_1, \dots, a_k \}$  is finite, then

$$RA = Ra_1 + Ra_2 + \dots + Ra_k$$

$RA$  is the submodule of  $M$  generated by  $A$ .

i.e. spanned by  $A$ .

③  $N \leq M$  is finitely generated if  $\exists$  finite  $A \leq M$  s.t.

$$N = RA.$$

④  $N \leq M$  is cyclic if  $N = Ra$  for some  $a \in M$ .

e.g.  $M = R$ . Submodules of  $R$  are ideals of  $R$ .

- finitely generated submodules of  $R$   
= finitely generated ideals of  $R$

- cyclic submodules of  $R$   
= principal ideals of  $R$ .

⚠ submodules of f.g. modules need not be f.g.  
e.g.  $M = R = \mathbb{Z}[x_1, x_2, \dots]$ .

$$(x_1, x_2, x_3, \dots) \leq M$$

↖ not finitely generated!

e.g.  $M = \mathbb{R}^n$  is finitely generated

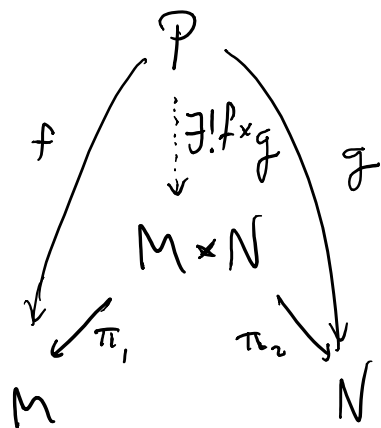
$$e_i = (0, \dots, 0, 1, 0, \dots, 0)$$

↑  $i$ -th coordinate

$$M = \mathbb{R}\{e_1, \dots, e_n\}$$

# Direct products, direct sums

$M, N, P$   $R$ -modules



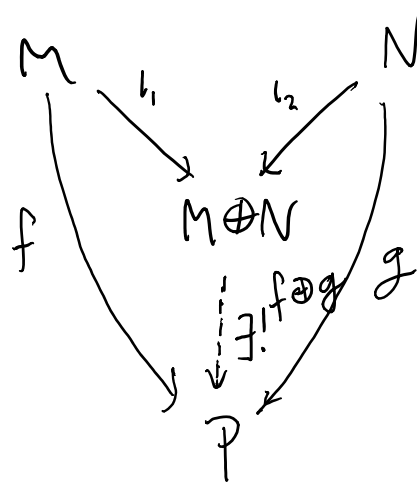
$M \times N$  is the product of  $M, N$  in  $R$ -modules.

$$(f \times g)(p) = (f(p), g(p))$$

Categories: Prefix "co" means flip all the arrows.

"Co-product" ??

direct sum



$\downarrow = \text{iota}$   
iota

$$M \oplus N = M \times N$$

$$i_1: M \rightarrow M \oplus N$$

$$m \mapsto (m, 0)$$

$$i_2: N \rightarrow M \oplus N$$

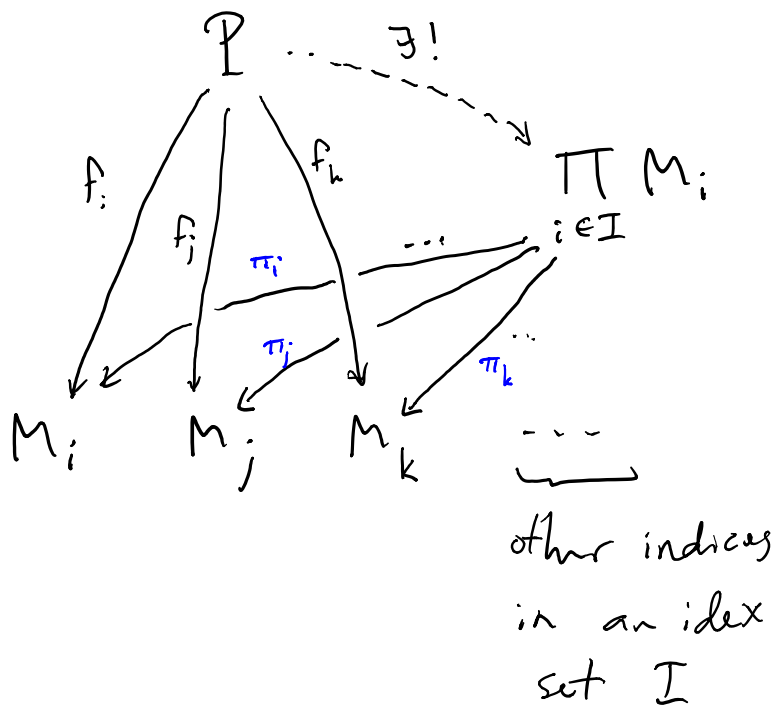
$$n \mapsto (0, n)$$

$$f \oplus g: M \oplus N \rightarrow P$$

$$(m, n) \mapsto f(m) + g(n)$$

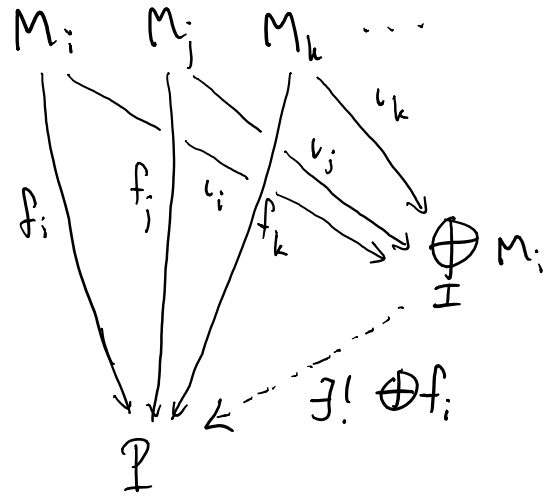
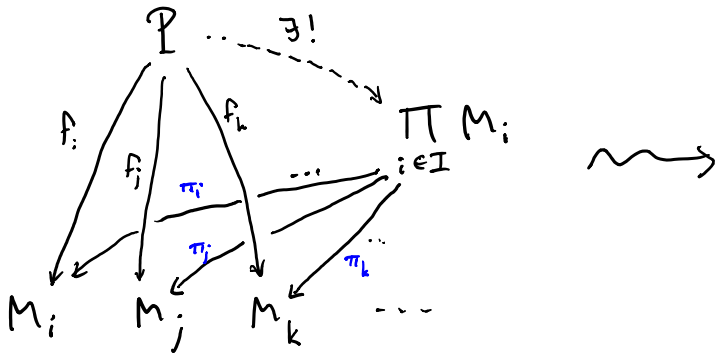
By induction,  $M_1 \times \dots \times M_n \cong M_1 \oplus \dots \oplus M_n$ .

Q What are products / coproducts indexed by an arbitrary set?



The product of  $M_i, i \in I$ , is a module  $\prod_I M_i$  equipped w/ "projection" homs  $\prod_I M_i \xrightarrow{\pi_j} M_j$  s.t.  $\forall R\text{-mod } P$  and collection of homs  $f_i : P \rightarrow M_i, i \in I,$

$$\exists! \prod_I f_i : P \longrightarrow \prod_I M_i \text{ s.t. } f_j = \pi_j \circ \prod_I f_i \quad \forall j \in I.$$



$$\prod_{i \in I} M_i = \left\{ (m_i)_{i \in I} \mid m_i \in M_i \right\}$$

$$\pi_j : \prod_{i \in I} M_i \rightarrow M_j$$

$$(m_i) \mapsto m_j$$

$$\prod_{i \in I} f_i : P \rightarrow \prod_{i \in I} M_i$$

$$p \mapsto (f_i(p))_{i \in I}$$

$$\bigoplus_{i \in I} M_i = \left\{ (m_i)_{i \in I} \mid m_i \in M_i, m_i = 0 \text{ for all but finitely many } i \in I \right\}$$

$$\iota_j : M_j \rightarrow \bigoplus_{i \in I} M_i$$

$$m_j \mapsto (m_i) \text{ where } m_i = \begin{cases} m_j & i=j \\ 0 & \text{o/w} \end{cases}$$

$$\bigoplus_{i \in I} f_i : \bigoplus_{i \in I} M_i \rightarrow P$$

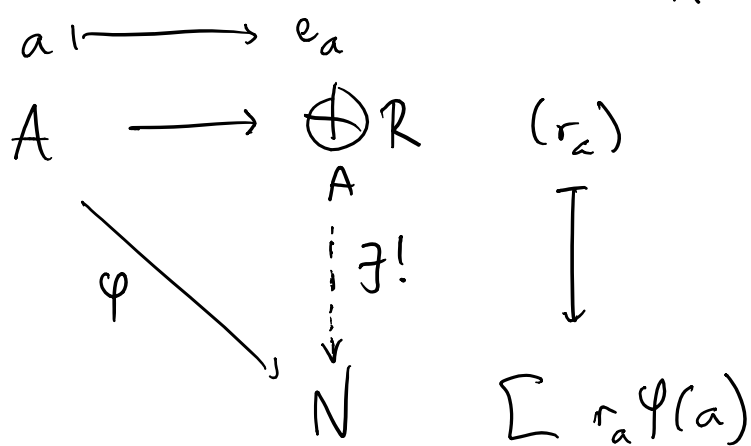
$$(m_i) \mapsto \sum_{i \in I} f_i(m_i)$$

well-defined  $\uparrow$  b/c almost every  $m_i = 0$ .

Def'n An  $R$ -module  $M$  is free on  $A \subseteq M$  if  $\forall x \in M$   
 $\exists! r_a \in R, a \in A$  s.t.  $x = \sum_{a \in A} r_a \cdot a$

Call  $A$  a basis of  $M$ .

When  $M$  is free on  $A$ ,  $M \cong \bigoplus_A R = R^A$



Thm  $\cdot \text{Hom}_R \left( \bigoplus_I M_i, N \right) \cong \prod_I \text{Hom}_R (M_i, N)$

$\cdot \text{Hom}_R \left( M, \prod_I N_i \right) \cong \prod_I \text{Hom}_R (M, N_i)$

as  $R$ -modules / abelian gps.