Lecture 41

Tuesday, April 14, 2015 10:06 AM

<u>Defn</u> R a ring, M, N R-modules

1. $\varphi : M \longrightarrow N$ is a <u>homomorphism</u> of R-modules if $\varphi(x+y) = \varphi(x) + \varphi(y)$ $\forall x, y \in M$ & $\varphi(rx) = r\varphi(x)$ $\forall r \in R, x \in M$.

2. φ is an isomorphism if it is also bijective (check: $\Longleftrightarrow$ $\exists$ 2-sided inverse hom)

3. $\ker(\varphi) = \{x \in M \mid \varphi(x) = 0\} = \varphi^{-1}(\{0\})$
   $\mathrm{im}(\varphi) = \varphi(M)$

4. $\mathrm{Hom}_R(M, N) = \{R\text{-mod homs } M \longrightarrow N\}$

<u>e.g.</u> · projection $\pi_i : R^n \longrightarrow R$ is a surjective
$(x_1, \dots, x_n) \longmapsto x_i$

hom w/ kernel $R^{i-1} \times 0 \times R^{n-i} = \{(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) \mid x_j \in R\}$

· R a field then R-mod homs are exactly R-linear transformation

- $\mathbb{Z}$-modules = abelian gps
  $\mathbb{Z}$-mod homs = homs of ab gps

- A an abelian group w/ a prime p s.t.
  $px = 0$ for all $x \in A$.
  Then A is a $\mathbb{Z}/p\mathbb{Z}$-module
  because $(p) = p\mathbb{Z}$ annihilates the $\mathbb{Z}$-module A
  Thus A is a $\mathbb{Z}/p\mathbb{Z}$-vector space $=$ $\mathbb{F}_p$-vector space.
  where $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is a field.

- What is $\mathrm{Hom}_{\mathbb{F}_p}(A, A) = \underbrace{\mathrm{End}}_{\text{endomorphisms}}{}_{\mathbb{F}_p}(A)$

$$\supseteq \mathrm{Aut}_{\mathbb{F}_p}(A) = \{\mathbb{F}_p\text{-lin trans } A \to A \text{ which are invertible}\}$$

$$= GL_{\mathbb{F}_p}(A)$$

If $A \cong \mathbb{F}_p^{\,n}$, $GL_{\mathbb{F}_p}(A) = GL_n(\mathbb{F}_p)$.

$1 \in R$

<u>Prop</u> M, N R-mods. $\varphi : M \longrightarrow N$ is a hom

$$\iff \varphi(x + ry) = \varphi(x) + r\varphi(y) \quad \forall r \in R,\ x, y \in M.$$

<u>Pf</u> $(\Rightarrow)$ If φ is a hom, $\varphi(x+ry) = \varphi(x) + \varphi(ry)$

$$= \varphi(x) + r\varphi(y).$$

$(\Leftarrow)$ If $r = 1$, $\varphi(x+y) = \varphi(x + 1\cdot y) = \varphi(x) + 1\cdot\varphi(y)$

$$= \varphi(x) + \varphi(y).$$

If $x = 0$, $\varphi(ry) = \varphi(0 + ry)$

$$= \varphi(0) + r\varphi(y)$$

$$= 0 + r\varphi(y)$$ [b/c φ is an add gp hom $M \to N$]

$$= r\varphi(y)$$

$\square$

<u>Prop</u> $\varphi, \psi \in \mathrm{Hom}_R(M, N)$ then $\varphi + \psi : M \longrightarrow N$, $m \longmapsto \varphi(m) + \psi(m)$ is an R-mod hom. In fact, this sum makes $\mathrm{Hom}_R(M, N)$ an abelian gp.

If R is commutative, then for $r \in R$, $r\varphi : M \longrightarrow N$, $m \longmapsto r \cdot \varphi(m)$ is an R-mod hom and this makes $\mathrm{Hom}_R(M,N)$ an R-module.

<u>Language</u> The category of R-mods is <u>enriched</u> in abelian gps (R-mods if R is comm).

<u>Pf of Prop</u>

$$(\varphi+\psi)(x+ry) = \varphi(x+ry) + \psi(x+ry)$$
$$= \varphi(x) + r\varphi(y) + \psi(x) + r\psi(y)$$
$$= (\varphi+\psi)(x) + r(\varphi+\psi)(y)$$

so $\varphi+\psi \in \mathrm{Hom}_R(M,N)$. The identity elt of $\mathrm{Hom}_R(M,N)$ is $0 : m \longmapsto 0$. The (additive) inverse of φ is $-\varphi : M \longrightarrow N$, $m \longmapsto -\varphi(m)$.

For R commutative, $r,s \in R$, $(r\varphi)(x+sy) =$

$$r\cdot\varphi(x+sy) = r(\varphi(x) + s\varphi(y)) = r\varphi(x) + rs\varphi(y)$$
$$= r\varphi(x) + sr\varphi(y) \qquad [\text{by commutativity}]$$
$$= (r\varphi)(x) + s(r\varphi)(y).$$

<u>Check</u> This indeed is a R-module structure. $\square$

Prop Compositions of R-mod homs are R-mod homs.
I.e. if $\varphi: M \to N$, $\psi: N \to P$ are R-mod homs, then $\psi \circ \varphi : M \to P$ is an R-mod hom.

Pf
$$\begin{aligned}(\psi\circ\varphi)(x+ry) &= \psi(\varphi(x+ry))\\ &= \psi(\varphi(x)+r\varphi(y))\\ &= \psi(\varphi(x)) + r\psi(\varphi(y))\\ &= (\psi\circ\varphi)(x) + r(\psi\circ\varphi)(y). \quad \square\end{aligned}$$

Prop Composition and + make $\mathrm{End}_R(M) = \mathrm{Hom}_R(M,M)$ into a ring (in fact an R-algebra when R is commutative).

Pf Exc. $\square$

Let N be a submodule of M. In particular, N is a (normal) subgp of $(M,+)$ and M/N makes sense as an abelian gp. In fact, for $r\in R$, $m \in M$
$$r(m+N) = rm + N$$ gives M/N the str of an R-module.

The proj'n map $\pi : M \longrightarrow M/N$ is an R-mod hom.

Indeed, $\pi(rm) = rm + N = r(m+N)$
$= r\,\pi(m)$.

<u>1st iso thm</u> $\varphi : M \longrightarrow N$ is an R-mod hom,

then $\ker(\varphi)$ is a submod of M & $M/\ker(\varphi) \cong \varphi(M)$.

<u>2nd — " —</u> A, B submods of M, $\underbrace{(A+B)}/B \cong A/A\cap B$

$\{a+b \mid a \in A, b \in B\} \subseteq M$

<u>3rd — " —</u> $A \subseteq B$, A, B submods of M then

$$(M/A)/(B/A) \cong M/B$$

<u>4th — " —</u> $\left\{ \begin{array}{c} \text{submods of } M \\ \text{containing } N \leq M \end{array} \right\} \overset{\cong}{\longleftrightarrow} \left\{ \begin{array}{c} \text{submods} \\ \text{of } M/N \end{array} \right\}$

$$A \supseteq N \longmapsto A/N$$

$$\pi^{-1}(\bar{A}) \longleftarrow\!\!\!\shortmid\ \bar{A}$$