Lecture 40

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R a (not necessarily commutative or w/ 1) ring.

<u>Defn</u> An <u>R-module</u> M is an abelian group $(M, +)$ equipped w/ an "action" $R \times M \longrightarrow M$

$$(r, m) \longmapsto rm$$

s.t. (1) $\forall r, s \in R$, $m \in M$, $(r+s)m = (rm) + (sm)$.

(2) $\forall r, s \in R$, $m \in M$, $(rs)m = r(sm)$

(3) $\forall r \in R$, $m, n \in M$, $r(m+n) = (rm) + (rn)$

If $1 \in R$, then (4) $1m = m$.

<u>Note</u> If R is a field (e.g. $\mathbb{R}$, $\mathbb{C}$) then a left R-module is the same as a vector space over R.

<u>e.g.</u>
- R is an R-module via mult'n in R
- $R^n = R^{\times n} = \underbrace{R \times R \times \cdots \times R}_{n \text{ times}}$ via

$$r \cdot (x_1, x_2, \ldots, x_n) = (rx_1, rx_2, \ldots, rx_n)$$

- $\mathbb{R}$ is an $\mathbb{R}$-module, but also $\mathbb{Q}$-module or a $\mathbb{Z}$-module since $\mathbb{Z}, \mathbb{Q}$ are subrings of $\mathbb{R}$.

- Consider R w/ its standard R-module structure. Submodules of R are precisely the ideals of R.

- If M is an R-module & $I \trianglelefteq R$ s.t. $IM = 0$, then M has the structure of an R/I-module via

$$(r+I)\cdot m = rm$$

<u>Ex</u>: Check the axioms in this case.

<u>E.G.</u> $\mathbb{Z}$-modules. If $(A,+)$ is an abelian gp, then A is a $\mathbb{Z}$-module via

$$na = \begin{cases} \underbrace{a+\cdots+a}_{n \text{ times}} & \text{if } n>0 \\ 0 & \text{if } n=0 \\ \underbrace{-a-a-\cdots-a}_{-n \text{ times}} & \text{if } n<0 \end{cases}$$

In fact, if $(M,+)$ has a $\mathbb{Z}$-module str, then $1\cdot m = m$ forces $2\cdot m = (1+1)m = m+m$, etc. so that all

all $\mathbb{Z}$-modules arise in this fashion

$$\{\mathbb{Z}\text{-modules}\} \longleftrightarrow \{\text{abelian groups}\}$$

<u>Defn</u> Let M be an R-module. Then $N \subseteq M$ is a submodule of M if $(N,+) \leq (M,+)$ and the restriction of the R-action on M to N has image in N, i.e. $R \cdot N \subseteq N$.

<u>Note</u> Submodules of a $\mathbb{Z}$-module are precisely subgps of the assoc ab gp.

<u>E.G.</u> $F[x]$-modules, F a field.

Let V be an F-vector space. Consider a linear transformation $T: V \longrightarrow V$.

Define $T^0 = id$, $T^n = T^{n-1} \circ T$ for $n > 0$.

i.e. $T^0 = id$, $T^1 = T$, $T^2 = T \circ T$, $T^3 = T \circ T \circ T$, ...

For $A, B : V \longrightarrow V$ linear transformations & $\alpha, \beta \in F$ define $\alpha A + \beta B : V \longrightarrow V$, $v \longmapsto \alpha A(v) + \beta B(v)$

$\alpha A + \beta B$ is a new linear transformation!

So for $p(x) \in F[x]$, $v \in V$

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$$

define $p(x)\, v = \left(a_n T^n + a_{n-1} T^{n-1} + \cdots + a_0 \right)(v)$

$$= a_n T^n(v) + a_{n-1} T^{n-1}(v) + \cdots + a_0 v$$

I.e. x acts as T and thus induces an $F[x]$-module structure on V.

Note The same F-v.s. V has a different $F[x]$-mod str for each $T : V \longrightarrow V$!

e.g.

- $T = 0$ then $p(x)\, v = a_0 v$
- $T = id$ then $p(x)\, v = \left(\sum a_k x^k \right) v$

$$= \left(\sum a_k \, id \right) v$$

$$= \sum a_k v$$

- $T : F^m \longrightarrow F^m$ shift $(x_1, \ldots, x_m) \mapsto (x_2, x_3, \ldots, x_m, 0)$

. $T: F^m \longrightarrow F^m$ shift $(x_1, \dots, x_m) \longmapsto (x_2, x_3, \dots, x_m, 0)$

Fact $\{F[x]\text{-modules}\} \overset{\cong}{\longleftrightarrow} \left\{\begin{array}{l} F\text{-vector spaces } V \\ \quad + \\ \text{linear } T: V \to V \end{array}\right\}$

$V \longmapsto$ V is an F-v.s. by action of const polys + T = action of x.

$$x^2 \cdot v = x(xv) = T(Tv) = T^2 v \quad \text{etc.}$$

Q What does a submodule of V look like?

We have $W \subseteq V$ a vector subspace of V s.t. $T(W) \subseteq W$. Such a subspace is called T-stable and the T-stable vector subspaces which are submodules of V.

e.g. For shift: if $(x_1, x_2, \dots, x_m) \in W$ then $(x_2, \dots, x_m, 0) \in W$. e.g. $F^k \times 0^{m-k}$

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E.G. R a comm ring w/ 1, G a finite gp

group ring $RG = \sum_{g \in G} a_g g$

Then RG-modules are representations of G.

so "representation theory" = the study of RG-modules.

Submodule Criterion

M an R-module, $N \subseteq M$ is a submodule

$\iff$ (1) $N \neq \emptyset$ & (2) $x + ry \in N$ $\forall x, y \in N, r \in R$

Pf ($\Rightarrow$) $0 \in N \neq \emptyset$ ✓

$x \in N, y \in N, r \in R \implies ry \in N$

$\implies x + ry \in N$

($\Leftarrow$) ✓ $\square$

Defn R comm ring w/1. An R-algebra is a ring A w/1 equipped w/a ring hom $f: R \to A$ s.t. $f(R) \subseteq Z(A)$.

e.g. · $F[x]$ is an F-alg via $f: F \to F[x]$, $a \mapsto a$

· $R \to RG$, $r \mapsto r \cdot 1$ makes RG an R-alg.