Lecture 37

Tuesday, April 7, 2015 10:03 AM

Thm R is a UFD $\iff$ $R[X]$ is a UFD.

Cor If R is a UFD, then $R[x_1, x_2, \dots, x_n]$ is a UFD.

Let R be an integral domain

Define $F = \mathrm{Frac}(R) = R \times (R \setminus \{0\}) / \sim$

where $(a,b) \sim (c,d)$ iff $ad = bc$.

Write $\frac{a}{b}$ for the equiv class of (a,b) in $\mathrm{Frac}(R)$.

Define $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$, $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

In fact possible to "invert" any multiplicative subset S of R — called "localization."

But you have to be careful if S contains 0 divisors: $(a,b) \sim (c,d) \iff \exists x \in R,\ x(ad-bc) = 0$.

Fact $(\mathrm{Frac}(R), +, \cdot)$ is a field, the smallest field containing R. ($R \hookrightarrow \mathrm{Frac}(R)$, $r \mapsto \frac{r}{1}$)

Gauss's Lemma Let R be a UFD, $F = \mathrm{Frac}(R)$, $p \in R[x] \subseteq F[x]$. If p is reducible in $F[x]$, then p is reducible in $R[x]$.

Pf Suppose $p = AB$, $A, B \in F[x]$. Multiply by a common multiple d of the denominators appearing in the coeffs of A, B:

$$(*) \quad dp = a'b', \quad a', b' \in R[x]$$

Note $d \in R \setminus 0$.

If $d \in R^\times$ then $p = (d^{-1}a')b'$ ~~reduces~~ factors p in $R[x]$.

If $d \notin R^\times$, factor $d = p_1 p_2 \cdots p_n$ into irreds in R.

Reduce $(*)$ mod p_1: $\underbrace{0 = \overline{a'}\,\overline{b'}}_{\text{eq'n in an integral domain}}$.

WLOG, $\overline{a'} = 0$. Thus $p_1 \mid a' \Rightarrow \dfrac{a'}{p_1} \in R[x]$.

So ⊛ transforms to

$$(p_2 p_3 \cdots p_n) \cdot p = \left(\frac{a'}{p_1}\right) \cdot b'$$

Thus induction implies the result. $\square$
on # of factors of d

<u>Cor</u> R a UFD, $F = \mathrm{Frac}(R)$, $p \in R[x]$. Suppose the gcd of coeffs of p is 1. Then p is irred in $R[x] \iff p$ is irred in $F[x]$.

<u>Pf</u> By Gauss's Lemma, p reducible in $F[x]$ $\Rightarrow$ p reducible in $R[x]$. If p is reducible in $R[x]$, then $p = ab$, a, b <u>nonconstant</u> irreds in $R[x]$. (b/c o/w an elt of R would divide all coeffs of p). Thus $p = ab$ is a factorization of p into nonunits of $F[x]$, b/c $F[x]^\times = F^\times$ so p is reducible in $F[x]$. $\square$

Thm R is a UFD $\iff$ $R[x]$ is a UFD.

Pf ($\Leftarrow$) Easy.

($\Rightarrow$) Let $p \in R[x]$ nonzero, not a unit.

Set $d = \gcd(\text{coeffs of } p)$.

So $p = dp'$ w/ $\gcd(\text{coeffs of } p') = 1$.

d factors uniquely into irreds in R which are automatically irreds in $R[x]$.

Thus WLOG, we can assume the gcd of the coeffs of p is 1 (replacing p w/ p').

Recall $F[x]$ is a Euclidean domain so p has a unique fact'n into irreds in $F[x]$.

By Gauss's lemma, p factors in $R[x]$ w/ factors F-multiples of factors in $F[x]$.

But the gcd of coeffs of $R[x]$ factors must be 1.

So, by the corollary to G's lemma, each factor is in fact irred in $R[x]$. Thus we have a fact'n into irreds!

Next: uniqueness of fact'n follows from uniqueness of fact'n in $F[x]$:

Have $p = p_1 p_2 \cdots p_n$ w/ p_i irred in $R[x]$.

But also know $p = q_1 \cdots q_n$ for q_i irred in $F[x]$, w/ same # of factors.

So if $p = p_1' p_2' \cdots p_m'$ is another fact'n into irreds in $R[x]$ then get $n = m$.

Get $p_1 = \frac{a}{b} \cdot p_1'$ etc.

Thus $b p_1 = a p_1'$

$\Rightarrow \gcd(\text{coeffs of } b) = \gcd(\text{coeffs of } a)$

But then $b = u \cdot a$, $u \in R^\times$

$$\Rightarrow p_1 = u^{-1} p_1', \; u^{-1} \in R^\times. \qquad \square$$

<u>Note</u> $\mathbb{Z}[x,y]$ is a UFD but <u>not</u> a PID.

Why not principal? (x, y) is not principal

In fact, $(2, x) \trianglelefteq \mathbb{Z}[x]$ is not principal.