Lecture 36

Monday, April 6, 2015 10:02 AM

Polynomials

$\boxed{R \text{ commutative ring w/ } 1 \neq 0}$

$$R[x] = \{ a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \mid n \in \mathbb{N},\ a_i \in R \}$$

$$= \{ (\ldots, 0, 0, a_n, a_{n-1}, \ldots, a_0) \mid a_i \in R,\ a_i = 0 \text{ for } i \gg 0 \}$$

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$. If $a_n \neq 0$, then $\deg(f(x)) := n$.

If $\deg(f(x)) = n$ & $a_n = 1$, we call $f(x)$ monic of degree n.

$$\left(\sum a_i x^i\right) + \left(\sum b_i x^i\right) = \sum (a_i + b_i) x^i$$

$$(\ldots, a_n, a_{n-1}, \ldots, a_0) + (\ldots, b_n, b_{n-1}, \ldots, b_0) = (a_i + b_i)_i$$

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$$\left(\sum a_i x^i\right)\cdot\left(\sum b_i x^i\right) = \sum_k \left(\sum_{i=0}^{k} a_i b_{k-i}\right) x^k$$

$$(a_i)\cdot(b_i) = \left(\sum_{i=0}^{k} a_i b_{k-i}\right)_k$$

$R[x]$ is also a commutative ring w/ $1 \neq 0$.

Canonical ring homomorphism $R \longrightarrow R[x]$, $a \longmapsto ax^0 = a$.

Consider $R \subseteq R[x]$ as the subring of constant polynomials.

Prop Suppose R is an integral domain.

① If p, q are nonzero elts of $R[x]$, then

$$\deg(pq) = \deg(p) + \deg(q).$$

[so $\deg(0) = -\infty$]

② $R[x]^\times = R^\times$

③ $R[x]$ is also an integral domain.

Pf of ② $R^\times \leq R[x]^\times$ b/c R a subring of $R[x]$.

If $pq = 1$ then $0 = \deg(p) + \deg(q) \Rightarrow \deg(p) = \deg(q) = 0$.

<u>Prop</u> $I \trianglelefteq R$ let $(I) \trianglelefteq R[x]$, i.e. $(I) = I[x]$.

Then $R[x]/(I) \cong (R/I)[x]$.

<u>Pf</u> Consider the surjective ring hom $\varphi : R[x] \longrightarrow (R/I)[x]$ which reduces coefficients mod I. The kernel of φ is (I). So by 1st iso thm, $R[x]/(I) \cong (R/I)[x]$.

$\square$

<u>Cor</u> If $I \trianglelefteq R$ is prime, then $(I) \trianglelefteq R[x]$ is prime as well.

<u>Pf</u> $R[x]/(I) \cong (R/I)[x] = \underbrace{(\text{int dom})[x]}_{\text{int domain.}}$ $\square$

<u>Universal Property</u> of <u>polynomial</u> rings

<u>If</u> $\varphi : R \to S$ is a ring hom and $s \in S$, then $\exists!$ ring hom $\psi : R[x] \longrightarrow S$ s.t. ① $\psi(x) = s$ and ② the diagram

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d] & S \\ R[x] \arrow[ur, dashed, "\exists!\psi", "x \mapsto s"'] & \end{tikzcd}$$

commutes.

Think: $R[x]$ is the "free commutative R-algebra on x".

Pf Trying to show

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d] & S \\ R[x] \arrow[ur, dashed, "\exists! \psi"'] & \end{tikzcd}$$

$x \mapsto s$

Take $\psi\left(\sum a_i x^i\right) = \sum \varphi(a_i) s^i$ ——

this is in fact the only ψ making the diagram commute w/ $\psi(x) = s$. $\square$

Of interest: take $\varphi = id_R$. Then for any $\alpha \in R$ we get the evaluation at α hom

$$ev_\alpha : R[x] \longrightarrow R$$

$$f(x) \longmapsto f(\alpha) = ev_\alpha f$$

$$\overset{\shortparallel}{\sum a_i x^i} \qquad\qquad \overset{\shortparallel}{\sum a_i \alpha^i}$$

Defn $\alpha \in R$ is a root of $f \in R[x]$ if

$f \in \ker ev_\alpha$. (I.e. $f(\alpha) = 0$!)

Defn $R[x_1, \dots, x_n] = \left(R[x_1, \dots, x_{n-1}]\right)[x_n]$.

<u>Fact</u> $R[x_1, \ldots, x_n] \cong R[x_{\sigma(1)}, \ldots, x_{\sigma(n)}]$

for any $\sigma \in S_n$.

In gen'l we can write any $f \in R[x_1, \ldots, x_n]$

as a sum of <u>monomials</u> $a x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}$

for $a \in R$, $d_i \in \mathbb{N}$.

$a \neq 0$
$\deg(a x_1^{d_1} \cdots x_n^{d_n}) = d_1 + d_2 + \cdots + d_n$.

$\text{mult.deg}(a x_1^{d_1} \cdots x_n^{d_n}) = (d_1, d_2, \ldots, d_n)$.

<u>Thm</u> F a field, $F[x]$ are a Euclidean domain via degree. In fact if $a(x), b(x) \in F[x]$ w/ $b(x) \neq 0$, then $\exists!\, q(x), r(x) \in F[x]$ s.t. $a(x) = q(x) b(x) + r(x)$ w/ $r(x) = 0$ or $\deg(r(x)) < \deg(b(x))$.

<u>Pf</u> If $a(x) = 0$, then $q(x) = r(x) = 0$.

If $a(x) \neq 0$, induce on $\deg(a) = n$.

If $n = 0$, $a(x) \in F^\times$ so take $q(x) = a^{-1}$, $r(x) = 0$.

If $n > 0$, let $m = \deg(b)$. If $n < m$, take $q(x) = 0$ & $r(x) = b(x)$. O/w $n \geq m$, write

$a(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$

$b(x) = b_m x^m + \dots + b_0$

$a'(x) = a(x) - \frac{a_n}{b_m} x^{n-m} b(x)$

$\deg(a'(x)) < \deg(a(x))$.

Thus $a' = q' b + r$ w/ $r = 0$ or $\deg(r) < \deg(b)$.

Let $q = q' + \frac{a_n}{b_m} x^{n-m}$. — this works.

Check uniqueness w/ deg. $\square$