Lecture 3

Wednesday, January 28, 2015 9:58 AM

The symmetric group

Let Ω be a set. Define $S_\Omega = \{f : \Omega \to \Omega \mid f \text{ is a bijection}\}$.

S_Ω has a group str via fn composition:

- $\circ$ is assoc
- $1 : \Omega \to \Omega$ is the identity fn $\omega \mapsto \omega$.
- every permutation of Ω has an inverse wrt $\circ$: $\sigma(a) = b$, $\sigma^{-1}(b) = a$.

symmetric group on Ω

Notation: For $n \in \mathbb{N}$, define $\underline{n} = \{1, 2, \dots, n\}$

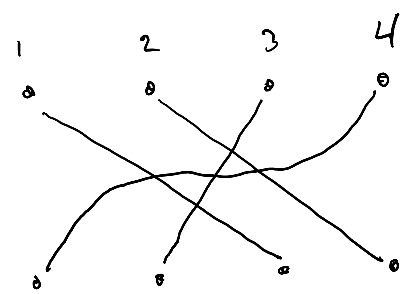
$$\underline{0} = \emptyset$$

$$S_{\underline{n}} = S_n = \{\text{bijections } \underline{n} \to \underline{n}\}$$

$\underline{4}$:

$\sigma \downarrow$

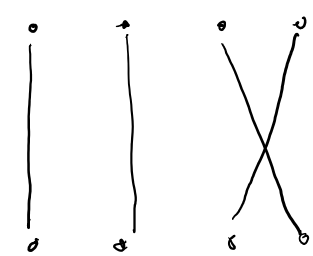
$\underline{4}$:



$\underline{4}$:

$\tau \downarrow$

$\underline{4}$:



Stack diagrams to compose.

Q How large is S_n?

How is a bij $\sigma : \underline{n} \to \underline{n}$ specified?

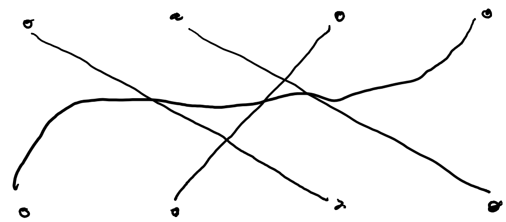
$\underline{n} \xrightarrow{\sigma} \underline{n}$	size
$1 \mapsto \sigma(1) \in \underline{n}$	n
$2 \mapsto \sigma(2) \in \underline{n} \setminus \sigma(\underline{1})$	$n-1$
$3 \mapsto \sigma(3) \in \underline{n} \setminus \sigma(\underline{2})$	$n-2$
$4 \mapsto \sigma(4) \in \underline{n} \setminus \sigma(\underline{3})$	$n-3$
$\vdots$	
$n-1 \mapsto \sigma(n-1) \in \underline{n} \setminus \sigma(\underline{n-2})$	2
$n \mapsto \sigma(n) \in \underline{n} \setminus \sigma(\underline{n-1})$	1

$$|S_n| = n(n-1)(n-2)\cdots 2\cdot 1 = n!$$

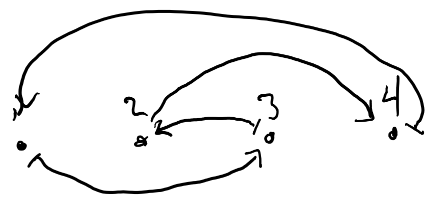
Note $|S_{\underline{0}}| = 1 = 0!$

Cycles A cycle is a string of integers representing an elt of S_n which cyclically permutes these integers.

$(1\ \ 3\ \ 2\ \ 4)$:



i.e.



What about?

<u>Lesson</u> Not every permutation is a cycle, but they are compositions of disjoint cycles!

Take $\sigma \in S_7$ s.t.

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 7 & 4 & 1 & 2 & 5 & 6 & 3 \end{array}$$

Travel through where 1 goes:

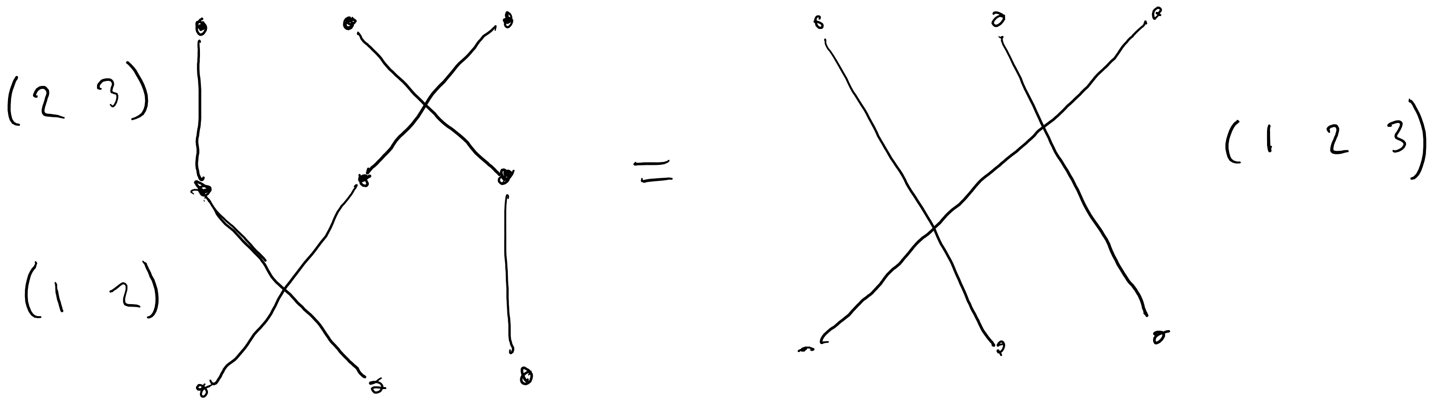
$$\sigma = (1\ \ 7\ \ 3)(2\ \ 4)(5)(6)$$
$$= (1\ \ 7\ \ 3)(2\ \ 4)$$

In general, any $\sigma \in S_n$ can be written

$$\sigma = (a_1\ a_2 \cdots a_{m_1})(a_{m_1+1}\ a_{m_1+2} \cdots a_{m_2}) \cdots (a_{m_{k-1}+1}\ a_{m_{k-1}+2} \cdots a_{m_k})$$

How do we compose cycles?

$$(1\ \ 2) \circ (2\ \ 3) = (1\ \ 2\ \ 3)$$



$$(2\ \ 3) \circ (1\ \ 2) = (1\ \ 3\ \ 2)$$

<u>Morals</u> ① S_n is nonabelian for $n \geqslant 3$.

② To compose cycles, re-implement cycle decomp algorithm.

<u>Q</u> How do we invert cycles?

$$(1\ \ 2\ \ 3)^{-1} = (3\ \ 2\ \ 1) = (1\ \ 3\ \ 2)$$

$$\begin{aligned}(1\ \ 3\ \ 2\ \ 4) &= (1\ \ 4\ \ 2\ \ 3)\\ &= (4\ \ 2\ \ 3\ \ 1)\end{aligned}$$

<u>A</u> Reverse order.

Fact Disjoint cycles commute.

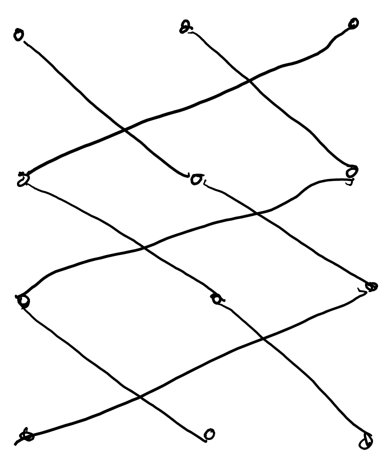
e.g. $(1\ 2\ 3)(4\ 5)$

$= (4\ 5)(1\ 2\ 3)$

Focus on $n=3$ case:

S_3 has order 6

	order
1	1
$(1\ 2)$	2
$(1\ 3)$	2
$(2\ 3)$	2
$(1\ 2\ 3)$	3
$(1\ 3\ 2)$	3

 $= 1 = (1\ 2\ 3)^3$

The **length** of a cycle is the number of integers it cyclically permutes. A cycle of length t is called a **t-cycle**.

Prop t-cycles have order t. ← disjoint

Exc If $\sigma = (2\text{-cycle})(3\text{-cycle})$ what is the order of σ?

Conj If σ is the product of disjoint cycles of lengths $t_1, t_2, \dots, t_k$, then

$$|\sigma| = \operatorname{lcm}(t_i).$$

Matrix / general linear groups.

F a field

Define $GL_n(F) = \{ n \times n \ F\text{-matrices } A \mid \det A \neq 0 \}$.

Claim $GL_n(F)$ is a gp under matrix mult.

- closed under matrix mult b/c

$$\det(AB) = (\det A)(\det B)$$

Quaternion group $Q_8 = \{ \pm 1, \pm i, \pm j, \pm k \}$

$\cdot$	1	i	j	k	-1	$\cdots$
1	1	i	j	k	-1	$\cdots$
i	i	-1	k	$-j$		
j	j	$-k$	-1	i		
k	k	j	$-i$	-1		
-1	-1					
$\vdots$	$\vdots$					

$\therefore$ (-1) commutes w/ everything