Lecture 27

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Presentations

Thought experiment: G a gp. What happens if we apply the universal property of $F(G)$ to $G \xrightarrow{id} G$?

$$\begin{tikzcd} G \arrow[r, hook] \arrow[dr, "id"'] & F(G) \arrow[d, "\exists! \pi"] \\ & G \end{tikzcd}$$

We get a surjective hom $F(G) \xrightarrow{\pi} G$.

In fact, if $S \subseteq G$ w/ $\langle S \rangle = G$, $S \hookrightarrow G$ produces a surj hom $F(S) \xrightarrow{\pi} G$ in the same way!

<u>Def'n</u> For $H \leq G$, let $\overline{H}$ = smallest normal subgroup of G containing H; this is called the <u>normal</u> <u>closure</u> of H.

<u>Def'n</u> $S \subseteq G$, $\langle S \rangle = G$.

1. A <u>presentation</u> for G is a pair (S, R) where $R \subseteq F(S)$ and $\overline{\langle R \rangle} = \ker(\pi : F(S) \longrightarrow G)$.
   S is the set of <u>generators</u>, R is the set of <u>relations</u>.
   We write $G = \langle S \mid R \rangle$.

2. G is called <u>finitely</u> <u>generated</u> if $\exists\, S \subseteq G$, $|S| < \infty$ w/ $G = \langle S \mid R \rangle$ for some R.
   G is called <u>finitely</u> <u>presented</u> if $\exists$ presentation $\langle S \mid R \rangle$ with both S, R finite.

<u>Shorthand</u> $\langle \{s_1, \dots, s_n\} \mid \{w_1, \dots, w_\ell\} \rangle = \langle s_1, \dots, s_n \mid w_1 = w_2 = \cdots = w_\ell = 1 \rangle$.

$$\langle S \mid w_1 w_2^{-1} \rangle = \langle S \mid \underbrace{w_1 = w_2}_{\text{equivalent to } w_1 w_2^{-1} = 1.} \rangle$$

<u>e.g.</u>

- $\mathbb{Z} \cong F(\{a\}) = \langle a \rangle$
- $\mathbb{Z} \times \mathbb{Z} \cong \langle a, b \mid [a,b] = 1 \rangle$ (recall $[a,b] = a^{-1}b^{-1}ab$)
- $\mathbb{Z}_n \cong \langle a \mid a^n = 1 \rangle$
- $\mathbb{Z}_n \times \mathbb{Z}_m \cong \langle a, b \mid a^n = b^m = 1 \rangle$.
- $D_{2n} \cong \langle r, s \mid r^n = s^2 = 1,\ s^{-1}rs = r^{-1} \rangle$
- $Q_8 \cong \langle i, j \mid i^4 = 1,\ j^2 = i^2,\ j^{-1}ij = i^{-1} \rangle$
- <u>HW</u> $S_n \cong \left\langle t_1, \dots, t_{n-1} \,\middle|\, \begin{array}{l} t_i^2 = 1,\ (t_i t_{i+1})^3 = 1, \\ [t_i, t_j] = 1 \text{ for } |i-j| \geq 2 \end{array} \right\rangle$

$(i\ \ i+1) \longleftarrow\!\shortmid\ t_i$

(proof by induction)

We can use presentations to determine homomorphisms b/w groups.

Suppose $G \cong \langle a, b \mid r_1, r_2, \ldots, r_k \rangle$.

Then we have the presentation sequence

$$\ker(\pi) \longrightarrow F(\{a,b\}) \xrightarrow{\pi} G$$

Any set map $\{a,b\} \longrightarrow H$, H some gp, induces $F(\{a,b\}) \xrightarrow{\rho} H$.

By the univ prop of quotients, we get an induced map $G \to H$

$$\begin{tikzcd} \ker(\pi) \arrow[r] \arrow[dr, "1"'] & F(\{a,b\}) \arrow[r] \arrow[d, "\rho"] & G \arrow[dl, dashed, "\exists!"] \\ & H & \end{tikzcd}$$

precisely when $\underbrace{\ker(\pi) \leq \ker(\rho)}$.

This is true precisely when $\rho(a), \rho(b)$ satisfy the relations of G!

If we play this game w/ $H = G$, get endomorphisms of G.

If we're careful, we can even determine automorphisms!