Lecture 26

Saturday, March 7, 2015 4:43 PM

Free Groups

Given a set S, want the group "freely generated" by S:
free of relations

Elements should be "words" in S: strings of elements of S along with inverses.

If $S = \{a, b, c\}$, expect a, ab, cab^{-1}, $aa^{-1}ba$ to be elements of the "free group" on S. Product works so that $ab \cdot a = aba$, $aa^{-1}ba = ba$, $cab^{-1} \cdot ba = cab^{-1}ba = caa$, etc.

If $F(S)$ is such an object, we also expect it to have the following universal property:

There is a natural inclusion $S \hookrightarrow F(S)$ and if $\phi : S \longrightarrow G$ is a function from S to a group G, then $\exists!$ hom $\Phi : F(S) \to G$ s.t.

$$\begin{tikzcd} S \arrow[r, hook] \arrow[dr, "\phi"'] & F(S) \arrow[d, "\exists! \Phi"] \\ & G \end{tikzcd}$$

commutes.

Note that any such object is unique: If $S \hookrightarrow F'(S)$ has the same property, then

$$\begin{tikzcd} S \arrow[r, hook] \arrow[dr, hook, bend right] \arrow[ddr, hook, bend right=50] & F(S) \arrow[d, "\exists!"] \\ & F'(S) \arrow[d, "\exists!"] \\ & F(S) \end{tikzcd}$$

$\exists!$, but must be $id_{F(S)}$ $\rightsquigarrow$ $F(S) \cong F'(S)$.

$$\begin{tikzcd} \arrow[r] \arrow[dr] & F'(S) \arrow[d, "\exists!"] \arrow[bend left]{} \\ & F(S) \end{tikzcd} \exists!$$, but must be $id_{F(S)}$ $\rightsquigarrow$ $F(S) \cong F'(S)$.

What about existence of $S \hookrightarrow F(S)$? Proceed by construction:

Let S^{-1} be a set disjoint from S equipped w/ a bijection $(\ )^{-1}: S \to S^{-1}$, $s \mapsto s^{-1}$.
Let $(\ )^{-1}$ also denote the inverse of the above map so that

$$(\ )^{-1}: S^{-1} \to S \quad \text{and} \quad (s^{-1})^{-1} = s.$$

Also consider a symbol 1 and define $1^{-1} = 1$.

<u>Defn</u> A <u>word</u> on S is a sequence $(s_1, s_2, s_3, \ldots)$ where $s_i \in S \cup S^{-1} \cup \{1\}$ and $s_i = 1$ for $\underbrace{i \gg 1}_{\text{"i sufficiently large"}}$.

A word on S is <u>reduced</u> if $s_{i+1} \neq s_i^{-1}$ for all i w/ $s_i \neq 1$ and if $s_k = 1$ for some k, then $s_i = 1$ for all $i \geq k$.

The reduced word $(1, 1, \ldots)$ is called the <u>empty word</u> and is denoted 1.

We denote $(s_1^{\varepsilon_1}, s_2^{\varepsilon_2}, s_3^{\varepsilon_3}, \ldots s_n^{\varepsilon_n}, 1, 1, \ldots)$ a reduced word w/ $s_i \in S$, $\varepsilon_i = \pm 1$ by $s_1^{\varepsilon_1} s_2^{\varepsilon_2} \cdots s_n^{\varepsilon_n}$.

Note that reduced words $s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n}$ and $r_1^{\delta_1} \cdots r_m^{\delta_m}$ are equal iff $m = n$, $s_i = r_i$, & $\varepsilon_i = \delta_i$.

Let $F(S) = \{\text{reduced words on } S\}$.

We define an operation on $F(S)$ by "juxtaposition" (or "concatenation") followed by "successive cancellation" of adjacent inverse elements.

<u>e.g.</u> $abc \cdot c^{-1}b^{-1}a = abcc^{-1}b^{-1}a = abb^{-1}a = aa = a^2$ ↑ new notation!

There is an unilluminating but perhaps comforting formula for this operation in the book

We define $S \hookrightarrow F(S)$ by $s \mapsto (s,1,1,\dots) = s$.

Thm $F(S)$ is a group under the above concatenate/cancel operation; moreover, $S \hookrightarrow F(S)$ satisfies the universal property.

Pf Identity: 1

Inverses: $(s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n})^{-1} = s_n^{-\varepsilon_n} \cdots s_1^{-\varepsilon_1}$.

Associativity: Surprisingly hard!

Moral Exc Prove by induction on word length or read the proof in the book.

Univ prop: For each set map $\phi : S \to G$ to a gp G we seek a unique hom Φ s.t.

$$\begin{tikzcd} S \arrow[r, hook] \arrow[dr, "\phi"'] & F(S) \arrow[d, dashed, "\Phi"] \\ & G \end{tikzcd}$$

commutes.

Clearly $\Phi(s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n}) = \phi(s_1)^{\varepsilon_1} \cdots \phi(s_n)^{\varepsilon_n}$ if Φ is a hom. (This handles uniqueness.) In fact, the above fmla makes Φ a hom, so we are done. $\square$

Defn $F(S)$ is the free group on S. A group F is a free group if it is isomorphic to $F(S)$ for some S.

The cardinality of S is called the free rank of $F(S)$.

Note Free group of free rank 0: $F(\emptyset) = 1$.

— " — 1: $F(\{a\}) \cong \mathbb{Z}$
$a \mapsto 1$

— " — 2: $F(\{a,b\})$ nonabelian, way bigger than $\mathbb{Z}^2$.