Lecture 25

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Semi-direct products

Recall $H, K \trianglelefteq G$, $H \cap K = 1 \Rightarrow HK \cong H \times K$.

What if $H \trianglelefteq G$, $K \leq G$, $H \cap K = 1$?

Still have $HK \leq G$ & $HK \xrightarrow{\varphi} H \times K$, $hk \mapsto (h,k)$ a well-defined bijection.

But it is no longer the case that φ is a hom.

$$(h_1k_1)(h_2k_2) = h_1k_1h_2(k_1^{-1}k_1)k_2$$
$$= \underbrace{h_1\underbrace{(k_1h_2k_1^{-1})}_{\in H}}_{h_3 \in H}\underbrace{(k_1k_2)}_{k_3 \in K} = h_3k_3$$

We seek to put a new group str on the set $H \times K$ so that $\varphi : HK \longrightarrow H \rtimes K$ is a gp iso. (Here $H \rtimes K$ is notation for $H \times K$ w/ new [to be determined] gp str.)

Clearly $(h_1, k_1) \cdot (h_2, k_2) = (h_1 \underbrace{(k_1h_2k_1^{-1})}_{\mathrm{conj}_{k_1}(h_2)}, k_1k_2)$

$\rtimes$ prod

is the desired fmla.

$\mathrm{conj}_{k_1} \in \mathrm{Aut}(H)$

In fact have $K \xrightarrow{\mathrm{conj}} \mathrm{Aut}(H)$ a hom.
$k \mapsto \mathrm{conj}_k$

Seek to gen'lize and permit any hom
$$\varphi : K \longrightarrow \mathrm{Aut}(H)$$
Such φ induces, of course, an action
$$K \underset{*}{\circlearrowright} H.$$
Write $k * h$ for $\varphi(k)(h)$.

<u>Def'n</u> Given groups H, K and a hom $\varphi: K \to \mathrm{Aut}(H)$ define $H \rtimes K$, the <u>semi-direct product</u> of H & K, to have underlying set $H \times K$ and mult'n

$$(h_1, k_1) \cdot (h_2, k_2) = (h_1 (k_1 * h_2), k_1 k_2)$$

where $k_1 * h_2 = (\varphi(k_1))(h_2)$.

<u>Thm</u> ① $H \rtimes K$ is a group of order $|H||K|$

② $H \to H \rtimes K$, $K \to H \rtimes K$ are inj homs
$h \mapsto (h, 1)$ $\quad k \mapsto (1, k)$

Thus it is reasonable to identify H w/ $H \rtimes 1$, K w/ $1 \rtimes K$ in $H \rtimes K$, in which case

③ $H \trianglelefteq H \rtimes K$

④ $H \cap K = 1$

⑤ $\forall h \in H, k \in K$, $khk^{-1} = k * h = \varphi(k)(h)$.

<u>Pf</u> ① Easy to check manually given that $K \circlearrowright_* H$ is a group action.

② The maps are clearly injective. To see the maps are homs, note that $(h, 1) \cdot (h', 1) = (h (1 * h'), 1) = (hh', 1)$

& $(1, k) \cdot (1, k') = (1, kk')$.

⑤ $(1, k)(h, 1)(1, k)^{-1} = (k * h, k)(1, k^{-1})$

$= ((k*h)(k*1), kk^{-1})$

$= (k * h, 1)$. I.e. $khk^{-1} = k * h$.

(4) Obvious.

(3) By (5), $khk^{-1} = k * h \in H$, so $K \leq N_{H \rtimes K}(H)$. Since $HK = H \rtimes K$ and $H \leq N_G(H)$ too, we get $H \rtimes K = HK \leq N_{H \rtimes K}(H)$ and thus $N_{H \rtimes K}(H) = H \rtimes K$, i.e. $H \trianglelefteq H \rtimes K$.

$\rtimes \neq \ltimes$. Always orient $\rtimes$ so that $\trianglelefteq$ & $\rtimes$ point in the same direction in Thm (3): $H \trianglelefteq H \rtimes K$.

$H \rtimes K$ depends on $\varphi : K \to \mathrm{Aut}(H)$, but we suppress φ from the notation. Perhaps it would be better to write $H \rtimes_\varphi K$?

<u>Prop</u> H, K gps, $\varphi : K \to \mathrm{Aut}(H)$ a hom. Then TFAE:

1. the identity function $H \rtimes K \to H \times K$ is a gp hom (and hence an iso)
2. φ is the trivial hom $K \to \mathrm{Aut}(H)$, $k \mapsto 1$
3. $K \trianglelefteq H \rtimes K$.

<u>Pf</u> (1) $\Rightarrow$ (2):

$$\begin{array}{ccc} (h_1, k_1)(h_2, k_2) & \longmapsto & (h_1, k_1)(h_2, k_2) \\ \| & & \| \\ (h_1(k_1 * h_2), k_1 k_2) & \longmapsto & (h_1 h_2, k_1 k_2) \end{array}$$

so $k_1 * h_2 = h_2 \Rightarrow *$ is the trivial action

$\Rightarrow \varphi$ is the trivial hom.

(2) $\Rightarrow$ (3) $\Rightarrow$ (1): <u>Moral exercise</u> $\square$

Examples

① p, q primes, $p < q$, $H = \mathbb{Z}_q$, $K = \mathbb{Z}_p$.

Recall If $p \nmid q-1$, then every gp of order pq is cyclic.

Note additionally that if $p \nmid q-1$, all homs $\mathbb{Z}_p \to Aut(\mathbb{Z}_q)$ are trivial.

Now assume $p \mid q-1$. By Cauchy's theorem, $Aut(\mathbb{Z}_q) \cong (\mathbb{Z}/q\mathbb{Z})^\times$ (which has order $q-1$) contains a subgp of order p.

Take $\varphi : K \to Aut(H)$ an inj hom w/ image this subgp.

The associated gp $G = H \rtimes K$ has order pq & K is not normal in G

HW G is the unique nonabelian gp of order pq.

② H an abelian gp, $K = \{\pm 1\}$, $\varphi : K \to Aut(H)$

$$-1 \longmapsto (h \mapsto h^{-1})$$

If $H = \mathbb{Z}_n$, then $H \rtimes K \cong D_{2n}$.

If $H = \mathbb{Z}$, then $H \rtimes K$ is called D_∞. $(D_{2\cdot\infty}?)$