Lecture 23

Wednesday, March 4, 2015 9:59 AM

① Functors & adjunctions (esp. for abelianization)

② Recognizing direct products.

① $(\ )^{ab} : \underline{Gp} \longrightarrow \underline{Ab}$

$\underline{Q}$ Given a hom $G \xrightarrow{f} H$ do we get a hom

$$\Downarrow$$

$$G^{ab} \xrightarrow[f^{ab}]{} H^{ab} \quad ?$$

$\underline{A}$

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d] \arrow[dr, "(*) = \mathrm{red} \circ f"] & H \arrow[d, "\mathrm{red}"] \\ G^{ab} \arrow[r, dashed, "\exists!\ \mathrm{hom}"', "f^{ab}"] & H^{ab} \end{tikzcd}
\qquad
\begin{tikzcd} G \arrow[r] \arrow[dr, "(*)"'] & G^{ab} \arrow[d, dashed, "\exists!"] \\ & A \ \text{ab gp} \end{tikzcd}$$

The assignment $G \longrightarrow G^{ab}$, $f \longmapsto f^{ab}$ is an example of a $\underline{\underline{\text{functor}}}$ (b/w categories).

$\underline{\text{Defn}}$ A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ b/w cats $\mathcal{C}, \mathcal{D}$ is

- an assigment $c \longmapsto F(c)$ for $c \in Ob\,\mathcal{C}$
- an assignment $(c \xrightarrow{f} c') \longmapsto (F(c) \xrightarrow{F(f)} F(c'))$

for $f \in \mathcal{C}(c, c')$

s.t. F preserves composition & identity morphisms:

$\forall$ composable pair of morphisms f, g in $\mathcal{C}$,

$$F(f \circ g) = F(f) \circ F(g)$$

and $F(id_c) = id_{F(c)}$ $\forall c \in Ob(\mathcal{C})$.

Prop $(\ )^{ab}$ is a functor $\underline{Gp} \longrightarrow \underline{Ab}$.

Pf By def'n, $(id_G)^{ab}$ is the unique hom making

$$\begin{tikzcd}
G \arrow[r, "id_G"] \arrow[d] & G \arrow[d] \\
G^{ab} \arrow[r, dashed, "(id_G)^{ab}"'] & G^{ab}
\end{tikzcd}$$

Note that $id_{G^{ab}}$ makes the diagram commute so $(id_G)^{ab} = id_{G^{ab}}$. ✓

Suppose $\varphi : G \longrightarrow H$, $\psi : H \longrightarrow K$ are gp homs.

$$\begin{tikzcd}
G \arrow[r, "\varphi"] \arrow[d] \arrow[rr, bend left, "\psi \circ \varphi"] & H \arrow[r, "\psi"] \arrow[d] & K \arrow[d] \\
G^{ab} \arrow[r, dashed, "\exists ! \varphi^{ab}"] \arrow[rr, dashed, bend right, "\exists ! (\psi \circ \varphi)^{ab}"'] & H^{ab} \arrow[r, dashed, "\exists ! \psi^{ab}"] & K^{ab}
\end{tikzcd}$$

Note $\psi^{ab} \circ \varphi^{ab}$ makes the outer diagram commute b/c the inner square commute and $\curvearrowright = \longrightarrow \circ \longrightarrow$.

By uniqueness, we must have $\psi^{ab} \circ \varphi^{ab} = (\psi \circ \varphi)^{ab}$.

Another functor: $U : \underline{Ab} \longrightarrow \underline{Gp}$.

In fact, $(()^{ab}, U)$ this is an adjoint pair of functors.

<u>Def'n</u> Functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \longrightarrow \mathcal{C}$ are <u>adjoint</u> (F <u>left adjoint</u> to G, G is <u>right adjoint</u> to F) if there is a natural bijection

$$\mathcal{C}(c, G(d)) \xleftrightarrow{\cong} \mathcal{D}(F(c), d).$$

<u>e.g.</u>

$$\underline{Gp}(G, U(A)) \cong \underline{Ab}(G^{ab}, A)$$
$$\varphi \longmapsto \bar{\varphi}$$

$$\begin{tikzcd} G \arrow[r] \arrow[dr, "\varphi"'] & G^{ab} \arrow[d, dashed, "\exists! \bar{\varphi}"] \\ & A \end{tikzcd} \qquad \longleftarrow\!\!\shortmid \ f$$

$f \circ \mathrm{red}$

$$\begin{tikzcd} G \arrow[r, "\mathrm{red}"] \arrow[dr, "f \circ \mathrm{red}"'] & G^{ab} \arrow[d, "f"] \\ & A \end{tikzcd}$$

<u>e.g.</u> In <u>Set</u>, let $\mathrm{Fun}(A,B) = \{\text{fns } A \to B\}$

$$\underline{Set}(X \times Y, Z) \cong \underline{Set}(X, \mathrm{Fun}(Y, Z))$$

$$X \times Y \xrightarrow{f} Z \longmapsto (x \longmapsto (f(x, -) : y \mapsto f(x,y)))$$

$$(x,y) \mapsto (g(x))(y) \longleftarrow\!\!\shortmid \ X \xrightarrow{g} \mathrm{Fun}(Y,Z)$$

Recognizing direct products

Recall $H, K \leq G$, $|HK| = \dfrac{|H||K|}{|H \cap K|}$

so if $H \cap K = 1$, then every $x \in HK$ has a unique representation as $x = hk$ for $h \in H$, $k \in K$.

Thm Suppose $H, K \trianglelefteq G$ & $H \cap K = 1$. Then $HK \cong H \times K$.
$hk \mapsto (h,k)$

Pf Know $HK \leq G$. Since $H \trianglelefteq G$, $\forall h \in H, k \in K$

$$k^{-1}hk \in H \implies \underbrace{h^{-1}k^{-1}hk}_{[h,k]} \in H$$

Note $[h,k] = \underbrace{(h^{-1}k^{-1}h)}_{\in K}\underbrace{k}_{\in K} \in K \cap H = 1$.

Thus $hk = kh$. By our recollection, we have a well-defined function $HK \longrightarrow H \times K$
$hk \longmapsto (h,k)$

Q Hom? $(hk)(h'k') \overset{\uparrow}{=} (hh')(kk') \longmapsto (hh', kk') = (h,k)\cdot(h',k')$

commutativity of H w/ K

A Yes.

$HK \to H \times K$ is a bij hom, thus an iso. $\square$

<u>Defn</u> $H, K \trianglelefteq G$, $H \cap K = 1$ we call HK the <u>internal direct product</u> of H & K.

<u>e.g.</u> $\mathbb{Z}_{pq} = \langle x \rangle$, p, q distinct primes.

$H = \langle x^q \rangle \qquad K = \langle x^p \rangle$

$|H| = p \qquad |K| = q$

$H \cap K = 1$

$\Rightarrow \mathbb{Z}_{pq} \cong H \times K \cong \mathbb{Z}_p \times \mathbb{Z}_q$.

Similarly, if $(m, n) = 1$, $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$.

<u>e.g.</u> $n = 2k+1$

$$D_{4n} \cong D_{2n} \times \mathbb{Z}_2$$