

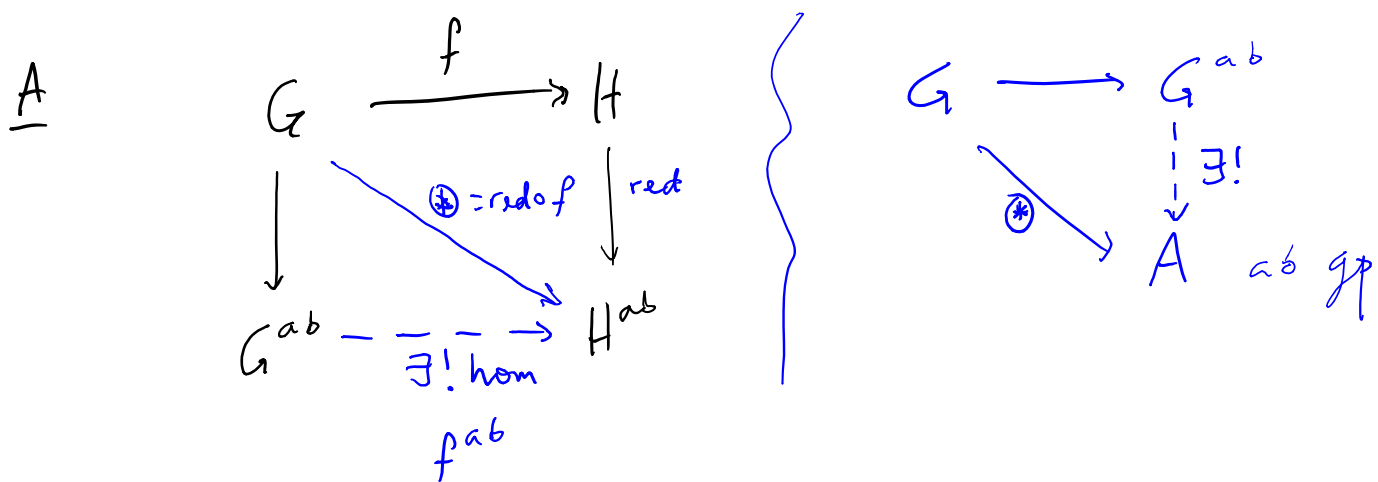
# Lecture 23

Wednesday, March 4, 2015 9:59 AM

- ① Functors & adjunctions (esp. for abelianization)
- ② Recognizing direct products.

①  $( )^{ab}: \underline{Grp} \longrightarrow \underline{Ab}$

Q Given a hom  $G \xrightarrow{f} H$  do we get a hom  $G^{ab} \xrightarrow{f^{ab}} H^{ab}$  ?



The assignment  $G \longrightarrow G^{ab}, f \longmapsto f^{ab}$  is an example of a functor (b/w categories).

Defn A functor  $F: \mathcal{C} \longrightarrow \mathcal{D}$  b/w cats  $\mathcal{C}, \mathcal{D}$  is

- an assignment  $c \longmapsto F(c)$  for  $c \in \text{Ob } \mathcal{C}$
- an assignment  $(c \xrightarrow{f} c') \longmapsto (F(c) \xrightarrow{F(f)} F(c'))$

for  $f \in C(c, c')$

s.t.  $F$  preserves composition & identity morphisms:

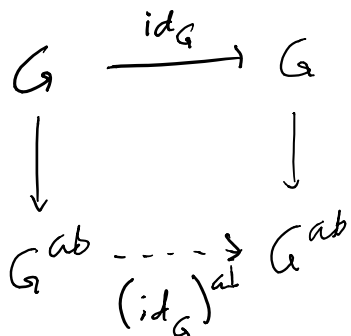
$\forall$  composable pair of morphisms  $f, g$  in  $C$ ,

$$F(f \circ g) = F(f) \circ F(g)$$

and  $F(id_c) = id_{F(c)} \quad \forall c \in Obj(C)$ .

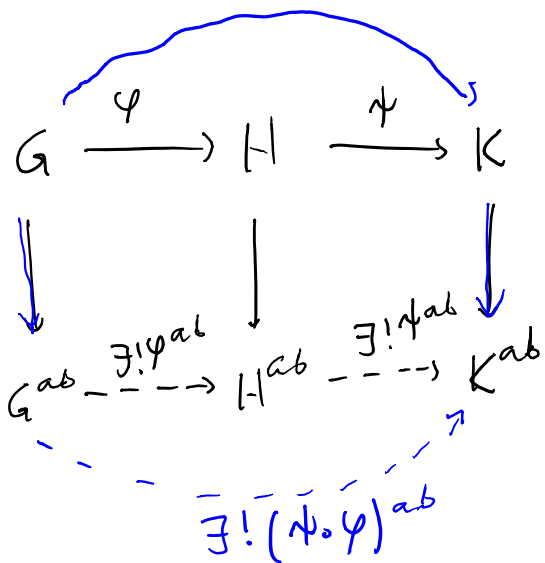
Prop  $( )^{ab}$  is a functor  $\underline{Grp} \rightarrow \underline{Ab}$ .

Pf By def'n,  $(id_G)^{ab}$  is the unique hom making



Note that  $id_{G^{ab}}$  makes the diagram commute so  $(id_G)^{ab} = id_{G^{ab}}$ . ✓

Suppose  $\varphi: G \rightarrow H, \psi: H \rightarrow K$  are gp homs,



Note  $\psi^{ab} \circ \varphi^{ab}$  makes the outer diagram commute b/c the inner square commutes and  $\psi \circ \varphi \xrightarrow{\quad} \psi^{ab} \circ \varphi^{ab}$ .

By uniqueness, we must have  $\psi^{ab} \circ \varphi^{ab} = (\psi \circ \varphi)^{ab}$ .

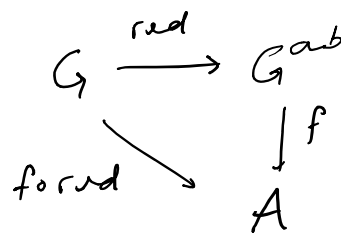
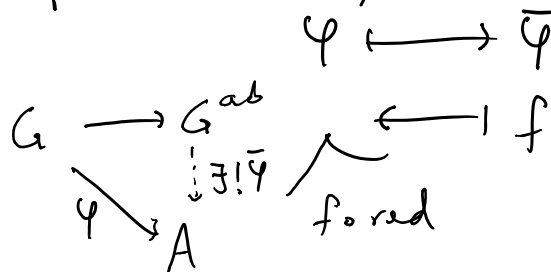
Another functor:  $U: \underline{Ab} \rightarrow \underline{Gp}$ .

In fact,  $(\ )^{ab}, U$  this is an adjoint pair of functors.

Def'n Functors  $F: \mathcal{C} \rightarrow \mathcal{D}$ ,  $G: \mathcal{D} \rightarrow \mathcal{C}$  are adjoint ( $F$  left adjoint to  $G$ ,  $G$  is right adjoint to  $F$ ) if there is a natural bijection

$$\mathcal{C}(c, G(d)) \xrightarrow{\cong} \mathcal{D}(F(c), d)$$

e.g.  $\underline{Gp}(G, U(A)) \cong \underline{Ab}(G^{ab}, A)$



e.g. In Set, let  $\text{Fun}(A, B) = \{\text{fns } A \rightarrow B\}$

$$\begin{aligned}
 \underline{\text{Set}}(X \times Y, Z) &\cong \underline{\text{Set}}(X, \text{Fun}(Y, Z)) \\
 X \times Y &\xrightarrow{f} Z \longmapsto (x \mapsto (f(x, -): y \mapsto f(x, y))) \\
 (x, y) &\mapsto (g(x))(y) \longleftarrow X \xrightarrow{F} \text{Fun}(Y, Z)
 \end{aligned}$$

# Recognizing direct products

Recall  $H, K \leq G$ ,  $|HK| = \frac{|H||K|}{|H \cap K|}$

so if  $H \cap K = 1$ , then every  $x \in HK$  has a unique representation as  $x = hk$  for  $h \in H, k \in K$ .

Thm Suppose  $H, K \trianglelefteq G$  &  $H \cap K = 1$ . Then  $HK \cong H \times K$ .  
 $hk \mapsto (h, k)$

Pf Know  $HK \leq G$ . Since  $H \trianglelefteq G$ ,  $\forall h \in H, k \in K$   
 $k^{-1}hk \in H \Rightarrow k^{-1}k^{-1}hk \in H$   
 "  $[h, k]$

Note  $[h, k] = \underbrace{(h^{-1}k^{-1}h)}_{\in K} \underbrace{k}_{\in K}$   $e \in K \cap H = 1$ .

Thus  $hk = kh$ . By our recollection, we have a well-defined function  $HK \rightarrow H \times K$   
 $hk \mapsto (h, k)$

Q Hom?  $(hk)(h'k') = (hh')(kk') \mapsto (hh', kk')$   
 $\uparrow$  commutativity of  $H \cup K$   $= (h, k) \cdot (h', k')$   
A Yes.

$HK \rightarrow H \times K$  is a bij hom, thus an iso.  $\square$

Defn  $H, K \leq G$ ,  $H \cap K = 1$  we call  $HK$  the internal direct product of  $H$  &  $K$ .

e.g.  $\mathbb{Z}_{pq} = \langle x \rangle$ ,  $p, q$  distinct primes.

$$H = \langle x^q \rangle \quad K = \langle x^p \rangle$$

$$|H| = p \quad |K| = q$$

$$H \cap K = 1$$

$$\Rightarrow \mathbb{Z}_{pq} \cong H \times K \cong \mathbb{Z}_p \times \mathbb{Z}_q$$

Similarly, if  $(m, n) = 1$ ,  $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$ .

e.g.  $n = 2k + 1$

$$D_{4n} \cong D_{2n} \times \mathbb{Z}_2$$