Lecture 22

Tuesday, March 3, 2015 10:04 AM

G a group, $x, y \in G$, $\emptyset \neq A, B \subseteq G$

Defn ① The commutator of x & y is

$$[x,y] = x^{-1}y^{-1}xy$$

② $[A,B] = \langle [a,b] \mid a \in A, b \in B \rangle \leq G$

the commutator subgp of A, B.

③ $[G,G] = G'$ is the commutator subgroup of G.

Prop ① $xy = yx[x,y]$ so $xy = yx \iff [x,y] = 1$.

Pf
$$yx[x,y] = yx\,x^{-1}y^{-1}xy$$
$$= yy^{-1}xy$$
$$= xy. \quad \square$$

② A subgroup $H \leq G$ is normal in $G \iff [H,G] \leq H$.

Pf $H \trianglelefteq G \iff \forall h \in H, g \in G, \; g^{-1}hg \in H$

$$\iff h^{-1}g^{-1}hg \in H$$

$$\iff [h,g] \in H \iff [H,G] \leq H. \; \square$$

(3) $\forall \sigma \in Aut(G)$, $\sigma([x,y]) = [\sigma(x), \sigma(y)]$.

Moreover, $[G,G]$ char $G \Rightarrow [G,G] \trianglelefteq G$,

$\uparrow$ H char G when $\sigma(H) = H$ $\forall \sigma \in Aut(G)$.

and $G/[G,G]$ is abelian.

Pf $\sigma([x,y]) = \sigma(x^{-1}y^{-1}xy) = \sigma(x^{-1})\sigma(y^{-1})\sigma(x)\sigma(y)$

$= \sigma(x)^{-1}\sigma(y)^{-1}\sigma(x)\sigma(y)$

$= [\sigma(x), \sigma(y)]$.

Thus $\sigma: \{[x,y] \mid x,y \in G\} \longrightarrow \{[x,y] \mid x,y \in G\}$

is bijective for each $\sigma \in Aut(G)$: $\sigma[x,y] = \sigma[z,w]$

$\sigma^{-1}(\sigma[x,y]) = \sigma^{-1}(\sigma[z,w])$

$[x,y] = [z,w]$

so injective.

$\sigma: [\sigma^{-1}x, \sigma^{-1}y] \longmapsto [x,y]$ so surjective.

Thus $\sigma([G,G]) = [G,G]$ $\forall \sigma \in Aut(G)$

so $[G,G]$ char G.

Suppose $x[G,G],\ y[G,G] \in G/[G,G]$

Then $(x[G,G])(y[G,G]) = (xy)[G,G]$

$= (yx[x,y])[G,G]$

$= (yx)[G,G]$ b/c $[x,y] \in [G,G]$.

$= (y[G,G])(x[G,G])$.

thus $G/[G,G]$ is abelian. $\square$

(4) $G/[G,G]$ satisfies the following universal property: If A is an abelian group and $\varphi: G \to A$ is a hom, then $\underbrace{\exists!\,\varphi^{ab}}_{\text{hom}} : G/[G,G] \to A$ s.t.

$$\begin{tikzcd} G \arrow[r] \arrow[dr, "\varphi"'] & G/[G,G] \arrow[d, "\exists!\,\varphi^{ab}"] \\ & A \end{tikzcd}$$

commutes.

In particular, if G/H is another abelian quotient of G, get $G/[G,G] \to G/H$ so $[G,G] \le H$.

Def'n $G/[G,G] = G^{ab}$ is the abelianization of G.

Note ④ tells us that G^{ab} is "initial" amongst abelian groups accepting homs from G.

Pf of ④

$$\begin{tikzcd} G \arrow[r] \arrow[dr, "\varphi"'] & G^{ab} \arrow[d, dashed, "\varphi^{ab}"] \\ & A \end{tikzcd}$$

a unique hom exists and makes the diagram commute $\iff [G,G] \leq \ker(\varphi)$

Check $\varphi([x,y]) = \varphi(x^{-1}y^{-1}xy)$

$= [\varphi(x), \varphi(y)] = 1$ b/c A is abelian.

This shows $[G,G] \leq \ker(\varphi)$. $\square$

e.g. • Let $G = Q_8$. What are the commutators in Q_8?

$1 = [1,1]$, $\quad [i,j] = (-i)(-j)ij$

$= +1 \cdot (ij)(ij)$

$= k \cdot k = -1$

Can check $[Q_8, Q_8] = \{\pm 1\}$

Thus $Q_8^{ab} = Q_8/\{\pm 1\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

- If G is abelian, $[G,G]=1$ and $G^{ab}=G$.

Facts
- $[S_n, S_n] = A_n \implies S_n^{ab} = S_n/A_n \cong \mathbb{Z}_2$.
- $[A_5, A_5] = A_5 \implies A_5^{ab} = A_5/A_5 = 1$.

Aside If $G^{ab}=1$, what homs $G \to A$, A abelian, exist?

$$
\begin{tikzcd}
G \arrow[r] \arrow[dr] & G^{ab} = 1 \arrow[d] \\
 & A
\end{tikzcd}
\qquad
\begin{tikzcd}
g \arrow[r, mapsto] \arrow[dr, mapsto, bend right] & 1 \arrow[d, mapsto] \\
 & 1
\end{tikzcd}
$$

Thus only 1 hom, the trivial hom.

Consider the cats
$$
\begin{tikzcd}
\underline{Gp} \arrow[r, shift left, "(\ )^{ab}"] & \underline{Ab} \arrow[l, shift left, "u"]
\end{tikzcd}
$$

$$G \longmapsto G^{ab}$$
$$u(A) = A \longleftarrow\!\shortmid\; A$$

$$
\underline{Gp}(G, A) \rightleftarrows \underline{Ab}(G^{ab}, A)
$$

$$\varphi \longmapsto \varphi^{ab}$$

$$f \circ (\text{red mod } [G,G]) \longleftarrow\!\shortmid\; f$$

$$G \xrightarrow{\text{red}} G/[G,G] = G^{ab} \xrightarrow{f} A$$

$$g \longmapsto gH$$
$$G \longrightarrow G/H$$
reduce mod H

$[G,G]\downarrow$

$G \xrightarrow{\text{red}} G/[G,G] = G^{ab} \xrightarrow{f} A$ reduce mod H

Get bijection $\underline{Gp}(G, U(A)) \xrightarrow{\approx} \underline{Ab}(G^{ab}, A)$

Example of an adjunction b/w functors.