

Lecture 22

Tuesday, March 3, 2015 10:04 AM

G a group, $x, y \in G$, $\emptyset \neq A, B \subseteq G$

Defn ① The commutator of x & y is

$$[x, y] = x^{-1}y^{-1}xy$$

$$② [A, B] = \langle [a, b] \mid a \in A, b \in B \rangle \leq G$$

the commutator subgroup of A, B .

$$③ [G, G] = G' \text{ is the } \underline{\text{commutator subgroup}} \text{ of } G.$$

Prop ① $xy = yx [x, y]$ so $xy = yx \Leftrightarrow [x, y] = 1$.

$$\begin{aligned} \underline{\text{Pf}} \quad yx[x, y] &= yx x^{-1}y^{-1}xy \\ &= yy^{-1}xy \\ &= xy. \quad \square \end{aligned}$$

② A subgroup $H \leq G$ is normal in $G \Leftrightarrow$

$$[H, G] \leq H.$$

$$\underline{\text{Pf}} \quad H \trianglelefteq G \Leftrightarrow \forall h \in H, g \in G, g^{-1}hg \in H$$

$$\Leftrightarrow h^{-1}g^{-1}hg \in H$$

$$\Leftrightarrow [h, g] \in H \Leftrightarrow [H, G] \leq H. \quad \square$$

$$(3) \quad \forall \sigma \in \text{Aut}(G), \quad \sigma([x, y]) = [\sigma(x), \sigma(y)].$$

Moreover, $[G, G] \text{ char } G \Rightarrow [G, G] \trianglelefteq G$,
 $\uparrow H \text{ char } G \text{ when } \sigma(H) = H$
 $\forall \sigma \in \text{Aut}(G)$.

and $G/[G, G]$ is abelian.

$$\begin{aligned} \text{pf } \sigma([x, y]) &= \sigma(x^{-1}y^{-1}xy) = \sigma(x^{-1})\sigma(y^{-1})\sigma(x)\sigma(y) \\ &= \sigma(x)^{-1}\sigma(y)^{-1}\sigma(x)\sigma(y) \\ &= [\sigma(x), \sigma(y)]. \end{aligned}$$

Thus $\sigma: \{[x, y] \mid x, y \in G\} \longrightarrow \{[x, y] \mid x, y \in G\}$

is bijective: $\sigma[x, y] = \sigma[z, w]$

for each $\sigma \in \text{Aut}(G)$

$$\sigma^{-1}(\sigma[x, y]) = \sigma^{-1}(\sigma[z, w])$$

$$[x, y] = [z, w]$$

so injective.

$\sigma: [\sigma^{-1}x, \sigma^{-1}y] \longmapsto [x, y]$ so surjective.

Thus $\sigma([G, G]) = [G, G] \quad \forall \sigma \in \text{Aut}(G)$

so $[G, G] \text{ char } G$.

Suppose $x[G, G], y[G, G] \in G/[G, G]$

$$\begin{aligned} \text{Then } (x[G, G])(y[G, G]) &= (xy)[G, G] \\ &= (yx[x, y])[G, G] \end{aligned}$$

$$= (yx)[G, G] \text{ b/c } [x, y] \in [G, G].$$

$$= (y[G, G])(x[G, G]).$$

thus $G/[G, G]$ is abelian. \square

(4) $G/[G, G]$ satisfies the following universal property:

If A is an abelian group and $\varphi: G \rightarrow A$ is a hom, then $\exists! \varphi^{ab}: G/[G, G] \rightarrow A$ s.t.

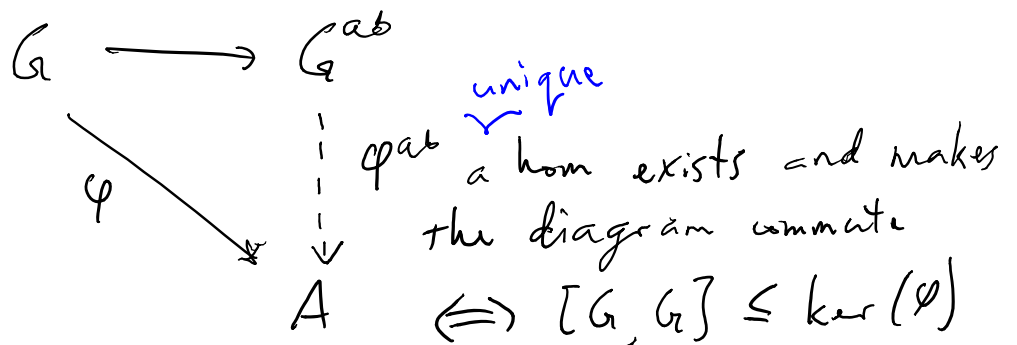
$$\begin{array}{ccc} G & \xrightarrow{\quad} & G/[G, G] \\ & \searrow \varphi & \downarrow \exists! \varphi^{ab} \\ & & A \end{array} \quad \text{commutes.}$$

In particular, if G/H is another abelian quotient of G , get $G/[G, G] \rightarrow G/H$ so $[G, G] \leq H$.

Defn $G/[G, G] = G^{ab}$ is the abelianization of G .

Note (4) tells us that G^{ab} is "initial" amongst abelian groups accepting homs from G .

Pf of (4)



check $\varphi([x, y]) = \varphi(x^{-1}y^{-1}xy)$
 $= [\varphi(x), \varphi(y)] = 1$ b/c A is abelian.

This shows $[G, G] \leq \ker(\varphi)$. \square

e.g. • Let $G = Q_8$. What are the commutators in Q_8 ?

$$1 = [1, 1], \quad [i, j] = (-i)(-j)ij$$

$$= +1 \cdot (ij)(ij)$$

$$= k \cdot k = -1$$

Can check $[Q_8, Q_8] = \{\pm 1\}$

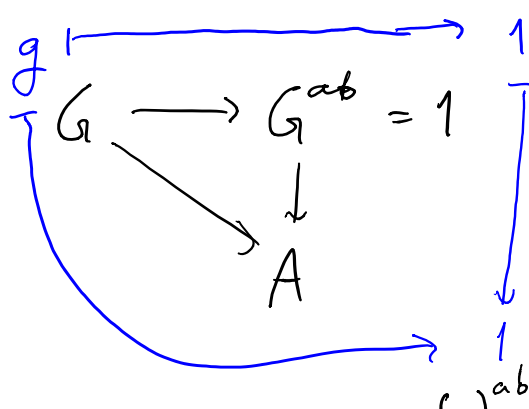
Thus $Q_8^{ab} = Q_8 / \{\pm 1\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

• If G is abelian, $[G, G] = 1$ and $G^{ab} = G$.

Facts • $[S_n, S_n] = A_n \implies S_n^{ab} = S_n/A_n \cong \mathbb{Z}_2$.

• $[A_5, A_5] = A_5 \implies A_5^{ab} = A_5/A_5 = 1$.

Aside If $G^{ab} = 1$, what homs $G \rightarrow A$, A abelian, exist?



Thus only 1 hom, the trivial hom.

Consider the cats $\underline{Grp} \begin{matrix} \xrightarrow{(\)^{ab}} \\ \xleftarrow{u} \end{matrix} \underline{Ab}$

$$G \xrightarrow{\quad} G^{ab}$$

$$u(A) = A \xleftarrow{\quad} A$$

$$\underline{Grp}(G, A) \xleftrightarrow{\quad} \underline{Ab}(G^{ab}, A)$$

$$\varphi \xrightarrow{\quad} \varphi^{ab}$$

$$f \circ (\text{red mod } [G, G]) \xleftarrow{\quad} f$$

$$\xleftarrow{\text{red}} G/H \xrightarrow{f} A$$

$$g \xrightarrow{\quad} gH$$

$$G \xrightarrow{\quad} G/H$$

↑
reduce mod H

$$G \xrightarrow{\text{red}} G/[G, G] = G^{ab} \xrightarrow{f} A \quad \begin{array}{c} \uparrow \\ \text{reduce mod } H \end{array}$$

Get bijection $\underline{\text{Grp}}(G, U(A)) \xrightarrow{\cong} \underline{\text{Ab}}(G^{\text{ab}}, A)$
Example of an adjunction b/w functors.