Lecture 21

Monday, March 2, 2015 9:59 AM

Products

In sets, what characterizes $A \times B$?

$$\begin{tikzcd} & C \arrow[dl, bend right, "f"'] \arrow[d, dashed, "h"] \arrow[dr, bend left, "g"] & \\ A & A \times B \arrow[l, "\pi_A"'] \arrow[r, "\pi_B"] & B \end{tikzcd}$$

$$\begin{tikzcd} & c \arrow[dl, bend right, mapsto] \arrow[d, mapsto] \arrow[dr, bend left, mapsto] & \\ f(c) & (f(c), g(c)) \arrow[l, mapsto] \arrow[r, mapsto] & g(c) \end{tikzcd}$$

Get $h : C \longrightarrow A \times B$

$c \longmapsto (f(c), g(c))$

$\pi_A h = f$

$\pi_B h = g$

But wait, there's more! h is unique

If $h' : C \longrightarrow A \times B$ makes the diagram commute, then $h = h'$.

Now suppose D is some other set equipped w/

$$\begin{tikzcd} & D \arrow[dl, "p_A"'] \arrow[dr, "p_B"] & \\ A & & B \end{tikzcd}$$

s.t. whenever $\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd}$ $\exists! h : C \longrightarrow D$ s.t. $p_A h = f$, $p_B h = g$

<u>Claim</u> Then $D \cong A \times B$

Pf

$$
\begin{tikzcd}
 & A\times B \arrow[d, dashed, "\exists! h"] \arrow[ddl, bend right, "\pi_A"'] \arrow[ddr, bend left, "\pi_B"] & \\
 & D \arrow[dl, "p_A"] \arrow[dr, "p_B"'] & \\
A & & B
\end{tikzcd}
\quad \& \quad
\begin{tikzcd}
 & D \arrow[d, dashed, "\exists! h'"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
 & A\times B \arrow[dl] \arrow[dr] & \\
A & & B
\end{tikzcd}
$$

where $p_A h = \pi_A$ $\quad \pi_A h' = p_A$

$p_B h = \pi_B$ $\quad \pi_B h' = p_B$

Observe: hh'

$$
\begin{tikzcd}
 & D \arrow[d, "hh'"] \arrow[ddl, bend right, "p_A"'] \arrow[ddr, bend left, "p_B"] & \\
 & D \arrow[dl, "p_A"] \arrow[dr, "p_B"'] & \\
A & & B
\end{tikzcd}
$$

makes this commute!

$p_A hh' = \pi_A h' = p_A$

$p_B hh' = \pi_B h' = p_B$

But $id : D \longrightarrow D$ also makes the diagram commute, so $hh' = id_D$.

Similarly, $h'h = id_{A\times B}$. Thus h, h' are inverse bijections!

For groups G, H is there a group w/ the same universal property (all fns replaced by gp homs)?

Natural guess: $G \times H$, $(g,h)\cdot(g',h') = (gg', hh')$

$$\begin{tikzcd} & G \times H \arrow[dl, "\pi_G"'] \arrow[dr, "\pi_H"] & \\ G & & H \end{tikzcd} \qquad \begin{tikzcd} & (g,h) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ g & & h \end{tikzcd}$$

Check π_G, π_H are homomorphisms. ✓

Now assume we have a gp A and homs f, g fitting into

$$\begin{tikzcd} & A \arrow[d, dashed, "\exists! h?"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\ & G \times H \arrow[dl, "\pi_G"] \arrow[dr, "\pi_H"'] & \\ G & & H \end{tikzcd}$$

Let's construct h: $h(a) = (f(a), g(a))$

Is h a hom? $h(ab) = (f(ab), g(ab))$

$= (f(a)f(b), g(a)g(b))$

$= (f(a), g(a))\cdot(f(b), g(b))$ ✓

h is unique b/c of the univ property for $\times$ of sets!

Note Same proof as before shows that if

$$\begin{tikzcd} & K \arrow[dl, "p_A"'] \arrow[dr, "p_B"] & \\ G & & H \end{tikzcd}$$

has the same universal property,

$K \cong G \times H$.

A category $\mathcal{C}$ consists of a collection of objects $\mathrm{Ob}\,\mathcal{C}$ and for each $A, B \in \mathrm{Ob}\,\mathcal{C}$, a morphism set $\mathcal{C}(A,B) = \mathrm{Hom}_{\mathcal{C}}(A,B)$ (which we think of as "arrows" $A \xrightarrow{f} B$, $f \in \mathcal{C}(A,B)$) and $\forall A, B, C \in \mathrm{Ob}\,\mathcal{C}$, $\circ : \mathcal{C}(B,C) \times \mathcal{C}(A,B) \longrightarrow \mathcal{C}(A,C)$ the composition of morphisms. These satisfy

1. composition associativity:
$$f \circ (g \circ h) = (f \circ g) \circ h$$
whenever this makes sense

2. identity: $\forall A \in \mathrm{Ob}\,\mathcal{C}$, $\exists\, id_A \in \mathcal{C}(A,A)$ s.t. $\forall f \in \mathcal{C}(A,B)$, $g \in \mathcal{C}(B,A)$,
$$f \circ id_A = f, \quad id_A \circ g = g$$

$$\begin{tikzcd} A \arrow[r, "f"] & B \\ A \arrow[u, "id_A"] \arrow[ur, "f"'] & \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "g"] \arrow[dr, "g"'] & A \arrow[d, "id_A"] \\ & A \end{tikzcd}$$

e.g. $\mathcal{C} = \underline{\text{Set}}$ $\text{Ob}\,\mathcal{C}$ = sets

$\mathcal{C}(A,B) = \{\text{fns } A \to B\}$

$\mathcal{C} = \underline{\text{Gp}}$ $\text{Ob}\,\mathcal{C}$ = groups

$\mathcal{C}(G,H) = \{\text{homomorphisms } G \to H\}$

similarly, $\underline{\text{Ab}}$ cat of ab gps

$\underline{\text{FinGp}}$ cat of finite groups

$\mathcal{C} = \underline{\text{Vect}}_k$ $\text{Ob}\,\mathcal{C}$ = vector spaces / field k

$\mathcal{C}(V,W) = \{\text{linear maps } V \to W\}$

$\mathcal{C} = \underline{\text{Field}}$ $\text{Ob}\,\mathcal{C}$ = fields

$\mathcal{C}(K,L) = \{\text{field homs } K \to L\}$

$\mathcal{C} = \underline{\text{Mat}}$ $\text{Ob}\,\mathcal{C} = \mathbb{Z}^+$

$m \times n = m+n$! $\mathcal{C}(m,n) = \{n \times m \text{ matrices}\}$

$\circ$ = matrix mult.

$\mathcal{C}(n,p) \times \mathcal{C}(m,n) \longrightarrow \mathcal{C}(m,p)$

Cartesian products of sets & direct product of gps are examples of <u>categorical</u> <u>products</u>.

For $A, B \in \mathrm{Ob}\,\mathcal{C}$, a triple (C, π_A, π_B) consisting of $C \in \mathrm{Ob}\,\mathcal{C}$, $\pi_A \in \mathcal{C}(C, A)$, $\pi_B \in \mathcal{C}(C, B)$ is the <u>product</u> of A & B if $\forall D \in \mathrm{Ob}\,\mathcal{C}$,

$f \in \mathcal{C}(D, A)$, $g \in \mathcal{C}(D, B)$, $\exists!\, h \in \mathcal{C}(D, C)$ s.t.

$$\begin{tikzcd} & D \arrow[ddl, bend right, "f"'] \arrow[d, "h"] \arrow[ddr, bend left, "g"] & \\ & C \arrow[dl, "\pi_A"] \arrow[dr, "\pi_B"'] & \\ A & & B \end{tikzcd}$$

commutes:

$\pi_B \circ h = g$

$\pi_A \circ h = f$.

Write $C = A \times B$ and frequently leave π_A, π_B out of our notation.

<u>Note</u> All but one of the e.g. cats have products

Note · If we have $A \xleftarrow{f} C \xrightarrow{g} B \in \mathcal{C}$

$$\updownarrow$$

$$C \to A \times B \quad \in \mathcal{C}$$

if $A \times B$ exists

· Notion of infinite products

⚠ <u>Fin Gp</u> doesn't have these.

<u>Thm</u> G, H gps, then $G \times H$ has a normal subgp $G \times 1 \trianglelefteq G \times H$ s.t. $G \times H / G \times 1 \cong H$.

<u>Pf</u> $G \times H \xrightarrow{\pi_H} H$ is a surj hom w/ kernel $G \times 1$. By 1^{st} iso thm, $H \cong G \times H / G \times 1$. $\square$