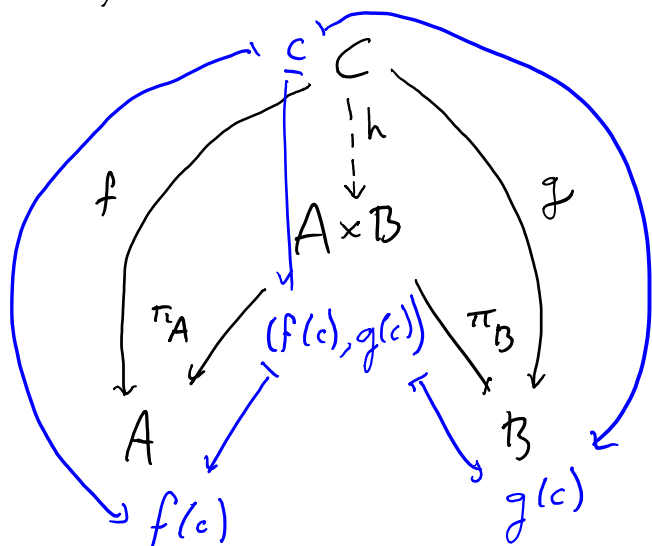


Lecture 21

Monday, March 2, 2015 9:59 AM

Products

In sets, what characterizes $A \times B$?



$$\text{Get } h: C \rightarrow A \times B \\ c \mapsto (f(c), g(c))$$

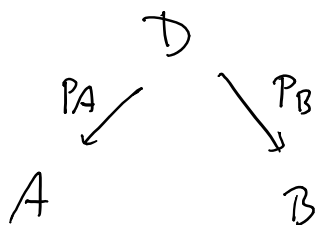
$$\pi_A h = f$$

$$\pi_B h = g$$

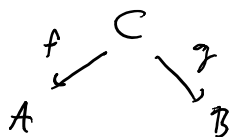
But wait, there's more! h is unique

If $h': C \rightarrow A \times B$ makes the diagram commute, then $h = h'$.

Now suppose D is some other set equipped w/



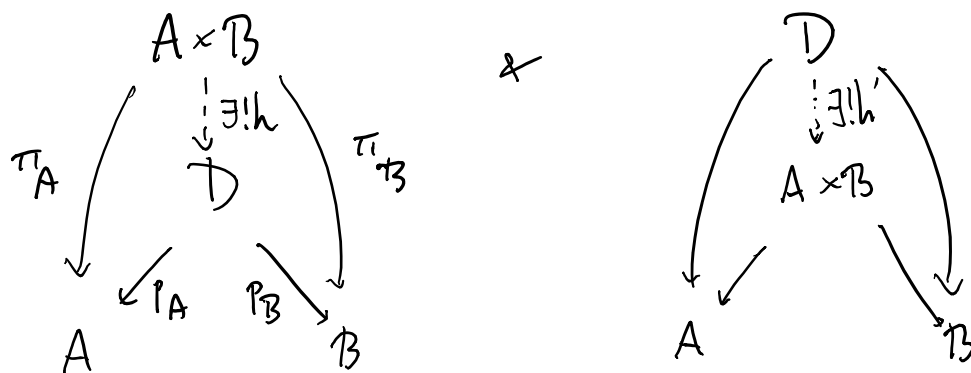
s.t. whenever



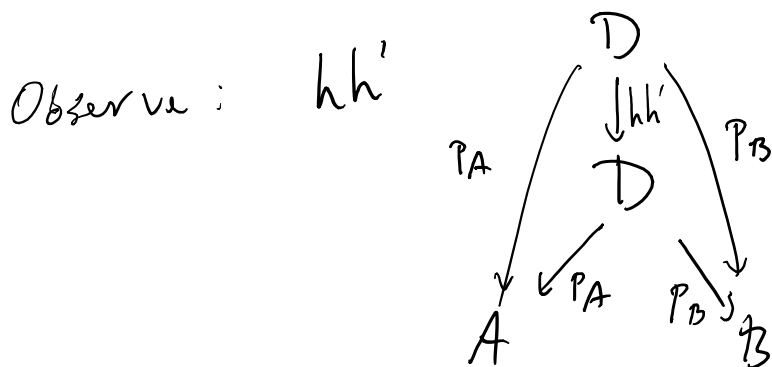
$$\exists! h: C \rightarrow D \text{ s.t. } p_A h = f \\ p_B h = g$$

Claim Then $D \cong A \times B$

Pf



where $p_A h = \pi_A$ $\pi_A h' = p_A$
 $p_B h = \pi_B$ $\pi_B h' = p_B$



makes this commute!

$$p_A hh' = \pi_A h' = p_A$$

$$p_B hh' = \pi_B h' = p_B$$

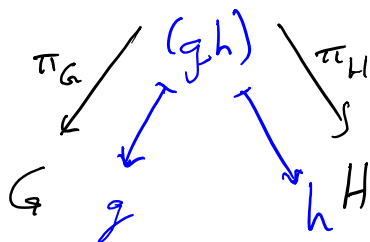
But $id : D \rightarrow D$ also makes the diagram commute,

so $hh' = id_D$.

Similarly, $h'h = id_{A \times B}$. Thus h, h' are inverse bijections!

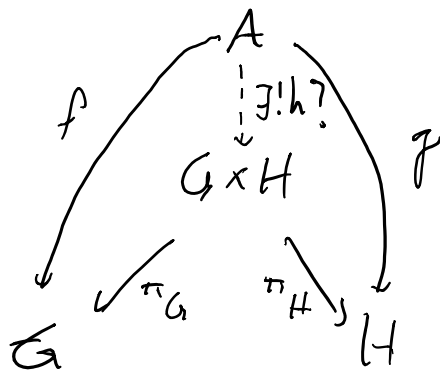
For groups G, H is there a group w/ the same universal property (all fns replaced by gp homs)?

Natural guess: $G \times H$, $(g, h) \cdot (g', h') = (gg', hh')$



Check π_G, π_H are homomorphisms. ✓

Now assume we have a gp A and homs f, g fitting into



Let's construct h : $h(a) = (f(a), g(a))$

Is h a hom? $h(ab) = (f(ab), g(ab))$

$$= (f(a)f(b), g(a)g(b))$$

$$= (f(a), g(a)) \cdot (f(b), g(b)) \quad \checkmark$$

h is unique b/c of the univ property for x of sets!

Note Same proof as before shows that if

$$\begin{array}{ccc} & K & \\ PA \swarrow & & \searrow PB \\ G & & H \end{array}$$

has the same universal property,
 $K \cong G \times H$.

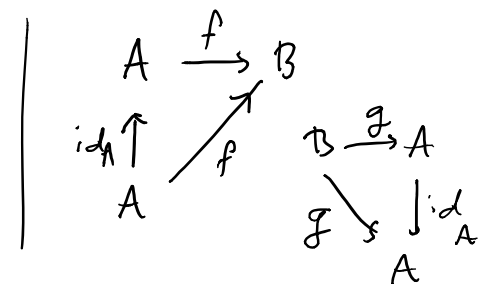
A category \mathcal{C} consists of a collection of objects $Ob \mathcal{C}$ and for each $A, B \in Ob \mathcal{C}$, a morphism set $\mathcal{C}(A, B) = Hom_{\mathcal{C}}(A, B)$ (which we think of as "arrows" $A \xrightarrow{f} B, f \in \mathcal{C}(A, B)$) and $\forall A, B, C \in Ob \mathcal{C}, \circ : \mathcal{C}(B, C) \times \mathcal{C}(A, B) \rightarrow \mathcal{C}(A, C)$ the composition of morphisms. These satisfy

① composition associativity:
 $f \circ (g \circ h) = (f \circ g) \circ h$
 whenever this makes sense

② identity: $\forall A \in Ob \mathcal{C}, \exists id_A \in \mathcal{C}(A, A)$ s.t.

$$\forall f \in \mathcal{C}(A, B), g \in \mathcal{C}(B, A),$$

$$f \circ id_A = f, \quad id_A \circ g = g$$



e.g. $\mathcal{C} = \underline{\text{Set}}$

$\text{Ob } \mathcal{C} = \text{sets}$

$\mathcal{C}(A, B) = \{\text{fns } A \rightarrow B\}$

$\mathcal{C} = \underline{\text{Grp}}$

$\text{Ob } \mathcal{C} = \text{groups}$

$\mathcal{C}(G, H) = \{\text{homomorphisms } G \rightarrow H\}$

similarly, Ab cat of ab grps

Fin Grp cat of finite groups

$\mathcal{C} = \underline{\text{Vect}}_k$

$\text{Ob } \mathcal{C} = \text{vector spaces / field } k$

$\mathcal{C}(V, W) = \{\text{linear maps } V \rightarrow W\}$

$\mathcal{C} = \underline{\text{Field}}$

$\text{Ob } \mathcal{C} = \text{fields}$

$\mathcal{C}(K, L) = \{\text{field homs } K \rightarrow L\}$

$\mathcal{C} = \underline{\text{Mat}}$

$\text{Ob } \mathcal{C} = \mathbb{Z}^+$

$m \times n = m + n$! $\mathcal{C}(m, n) = \{n \times m \text{ matrices}\}$

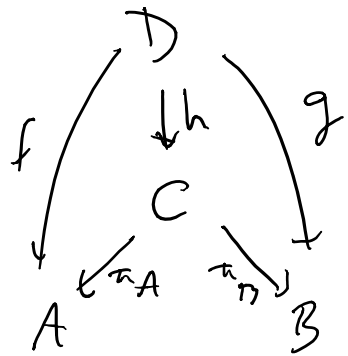
$\circ = \text{matrix mult.}$

$\mathcal{C}(n, p) \times \mathcal{C}(m, n) \rightarrow \mathcal{C}(m, p)$

Cartesian products of sets & direct product of gps are examples of categorical products.

For $A, B \in \text{Ob } \mathcal{C}$, a triple (C, π_A, π_B) consisting of $C \in \text{Ob } \mathcal{C}$, $\pi_A \in \mathcal{C}(C, A)$, $\pi_B \in \mathcal{C}(C, B)$ is the product of A & B if $\forall D \in \text{Ob } \mathcal{C}$,

$f \in \mathcal{C}(D, A)$, $g \in \mathcal{C}(D, B)$, $\exists ! h \in \mathcal{C}(D, C)$ s.t.



commutes:

$$\pi_B \circ h = g$$

$$\pi_A \circ h = f$$

Write $C = A \times B$ and frequently leave π_A, π_B out of our notation.

Note All but one of the e.g. cats have products

Note . If we have $A \xleftarrow{f} C \xrightarrow{g} B \quad e \in \mathcal{C}$

$$\begin{array}{c} \updownarrow \\ C \\ \downarrow \\ A \times B \end{array} \quad e \in \mathcal{C}$$

if $A \times B$ exists

- Notion of infinite products
- ↳ Fin Gr doesn't have these.

Thm G, H gpi, then $G \times H$ has a normal subgroup $G \times 1 \trianglelefteq G \times H$ s.t. $G \times H / G \times 1 \cong H$.

Pf $G \times H \xrightarrow{\pi_H} H$ is a surj hom w/ kernel $G \times 1$. By 1st iso thm, $H \cong G \times H / G \times 1$. \square