Lecture 20

Friday, February 27, 2015 10:05 AM

<u>Prop</u> If G is a group of order 30, then $\exists N \trianglelefteq G$ w/ $N \cong \mathbb{Z}_{15}$.

<u>Pf</u> Any $N \leq G$ of order 15 is of index 2 and thus automatically normal.

If $[G:H]=2$, then

$G/H = \{H, gH\}$

$H\backslash G = \{H, Hg'\}$.

Suffices to show $gH = Hg'$.

Since $G = H \amalg gH$, $gH = G \setminus H$. Similarly, $Hg' = G \setminus H$. ✓

Moreover, $15 = 3 \cdot 5$.

Since $3 \nmid 5-1=4$, we know any such gp is cyclic of order 15.

Let $P \in Syl_5(G)$, $Q \in Syl_3(G)$. $P \cap Q = 1$.

If $P \trianglelefteq G$ or $Q \trianglelefteq G$, then

$$|PQ| = \frac{|P||Q|}{|P\cap Q|} = 15.$$

Assume $P, Q \ntrianglelefteq G$. Then $n_5 = 6$, $n_3 = 10$ by Sylow 3.

($n_p \equiv 1 \pmod p$, $n_p \mid m$ where $|G| = p^\alpha \cdot m$, $p \nmid m$.)

Elts of order 5: $4 \cdot 6 = 24$

" — " 3: $2 \cdot 10 = \underline{20}$

44 elts of a group of order 30. $\square$

Lemma $H, K \leq G$, $|H| = p$, $|K| = q$, $p \neq q$ prime. Then $H \cap K = 1$.

Pf Assume for $\exists x \neq 1 \in H \cap K$. Since $x \in H$, $|x| \,\big|\, p \Rightarrow |x| = p$. Similarly, $|x| = q$. $\square$

Lemma $H, H' \leq G$, $|H| = p = |H'|$, then either $H \cap H' = 1$ or $H = H'$. $\square$

<u>Thm</u> A_n is simple for $n \geq 5$.

<u>Pf</u> by induction on n. We have already shown A_5 is simple. Assume $n \geq 6$. Let $G = A_n$.

$G \circlearrowright \underline{n} = \{1, 2, \dots, n\}$. Let G_i = isotropy gp of i for $i \in \underline{n}$. Then $G_i \cong A_{n-1}$ for any $i \in \underline{n}$.

By induction hypothesis, G_i is simple for $i \in \underline{n}$.

Assume for $\cancel{\text{?}}$ that $\exists H \trianglelefteq G$ s.t. $1, G \neq H$.

We'll first show that $\forall \tau \in H \setminus \{1\}$, $\tau(i) \neq i$ for any $i \in \underline{n}$.

Assume for $\cancel{\text{?}}$ $\exists \tau \in H \setminus \{1\}$ s.t. $\tau(i) = i$ for some $i \in \underline{n}$.

Then $\tau \in G_i \cap H \trianglelefteq G_i \Rightarrow G_i \cap H = G_i \Rightarrow G_i \leq H$.

Now for $\sigma \in G$, $\sigma G_i \sigma^{-1} = G_{\sigma(i)}$ [check]

so $\forall i$, $\sigma G_i \sigma^{-1} \leq \sigma H \sigma^{-1} = H$

i.e. $G_{\sigma(j)} \leq H$ $\forall j \in \underline{n}$.

Take $\lambda \in A_n$ and write it as a product of an even # of transpositions, i.e. $\lambda = \lambda_1 \lambda_2 \cdots \lambda_t$ where each λ_i is the product of 2 transpositions.

Each λ_k fixes $\geq n-4 \geq 2$ elts of $\underline{n}$

$\shortparallel$

$(a\ b)(c\ d)$

Thus $\lambda_k \in G_j$ for some $j \in \underline{n}$.

$$G = \langle G_1, G_2, \dots, G_n \rangle \leq H \quad \cancel{\text{☺}}$$

[$\langle \{3\text{-cycles}\} \rangle = G = A_n$

Every 3-cycle is in some G_i

b/c $n \geq 4$

(another proof)]

So any $\tau \in H \setminus \{1\}$ fixes nothing in $\underline{n}$, i.e.

$\tau \notin G_i \quad \forall i \in \underline{n}$.

It follows that if $\tau_1, \tau_2 \in H$ and $\tau_1(i) = \tau_2(i)$ for some i, then $\tau_1 = \tau_2$. [look at $\tau_2^{-1}\tau_1(i) = i$

$\Rightarrow \tau_2^{-1}\tau_1 = 1$]

<u>Claim</u> $\forall \tau \in H$, the cycle decomposition of τ only contains 2-cycles.

For $\cancel{\text{☺}}$, assume $\tau \in H$, $\tau = (a_1\ a_2\ a_3 \dots)(b_1\ b_2 \dots) \dots$ in its cycle decomp.

Take $\sigma \in G$ s.t. $\sigma(a_1) = a_1$, $\sigma(a_2) = a_2$, $\sigma(a_3) \neq a_3$.

(σ exists b/c $n \geq 5$: $\sigma = (a_3\ b\ c)$, $b, c \in \underline{n} \setminus \{a_1, a_2\}$.)

Then $\tau_1 = \sigma \tau \sigma^{-1} = (\sigma(a_1)\ \sigma(a_2)\ \sigma(a_3)\ \cdots)(\sigma(b_1)\ \sigma(b_2) \cdots) \cdots$

$$= (a_1\ a_2\ \sigma(a_3) \cdots) \cdots$$

$\Rightarrow \tau \neq \tau_1 \in H$ and $\tau(a_1) = \tau_1(a_1) = a_2$ $\lightning$.

We now know $\tau \in H \setminus \{1\}$, then

$$\tau = (a_1\ a_2)(a_3\ a_4)(a_5\ a_6) \cdots$$

Take $\sigma = (a_1\ a_2)(a_3\ a_5) \in G$

then $\tau_1 = \sigma \tau \sigma^{-1} = (\sigma(a_1)\ \sigma(a_2))(\sigma(a_3)\ \sigma(a_4))$
$(\sigma(a_5)\ \sigma(a_6)) \cdots$

$$= (a_1\ a_2)(a_5\ a_4)(a_3\ a_6) \cdots$$

Then $\tau, \tau_1 \in H$ and $\tau(a_1) = \tau_1(a_1) = a_2$. $\lightning$.

Thus $H \trianglelefteq G \Rightarrow H = 1$ or G. I.e. G is simple.

$\square$