Lecture 2

Tuesday, January 27, 2015 9:56 AM

Notation $a \cdot b = ab$

$$a^n = \underbrace{a \cdot a \cdots a}_{n \text{ times}} \quad n > 0$$

$$a^{-n} = (a^{-1})^n \quad n > 0$$

$$a^0 = e = 1$$

Write G for $(G, \cdot)$ and this is a group.

Prop $\forall a, b \in G$, the equations

① $ax = b$

② $ya = b$

have unique sol'ns $x, y \in G$.

Pf Multiply ① on the left by a^{-1}:

$$a^{-1} \cdot (ax) = a^{-1} \cdot b$$

$$\Rightarrow \quad x = a^{-1} b.$$

Multiply ② by a^{-1} on the right:

$$y = b a^{-1}. \qquad \square$$

Cor [Cancellation] $\forall a \in G$

1. $au = av \Rightarrow u = v$
2. $ua = va \Rightarrow u = v$.

Cor If $ab = 1$, then $a = b^{-1}$ & $b = a^{-1}$.

Note $|G|$ is its coarsest — but still very important — invariant.

cardinality of the set G

Defn The **order** of an element $x \in G$ is $|x|$, the smallest positive integer s.t.

$$x^{|x|} = 1.$$

e.g. $\mathbb{Z}_4 \ni 1$. What is $|1|$?

$$1 \cdot 1 = 1$$

$$2 \cdot 1 = 1 + 1 = 2$$

$$3 \cdot 1 = 1 + 1 + 1 = 3$$

$$4 \cdot 1 = 1 + 1 + 1 + 1 = 0$$

What if no such $|x|$ exists?

In this case, declare $|x| = \infty$.

$$| \; | : G \longrightarrow \underbrace{\mathbb{Z}^+}_{\{1,2,3,\ldots\}} \cup \{\infty\}$$

Multiplication table

$(\mathbb{Z}_3, +)$

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

·	1	x	y	z
1	1	x	y	z
x	x	1	z	y
y	y	z	1	x
z	z	y	x	1

abelian group isomorphic to $\{\pm 1\} \times \{\pm 1\}$

Dihedral groups

Record symmetries of regular n-gons P_n

Consider the rigid motions (in $\mathbb{R}^3$) taking P_n to itself; call it D_{2n}.

This forms a group under composition:

$s, t \in D_{2n}$

$$\begin{tikzcd} P_n \arrow[r, "t"] \arrow[dr, "st"'] & P_n \arrow[d, "s"] \\ & P_n \end{tikzcd}$$

Our first commutative diagram!

Q How large is D_{2n}?

$n=5$: 5 rotations, 5 reflections $\Big\}$ 10 elts

conj $|D_{2n}| = 2n$.

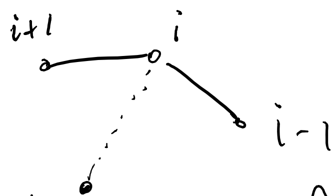
Pf of conj

① Observe that a symmetry of P_n is determined by where it sends $1 — 2$

② How many places could $1 — 2$ possibly go?

- By rotation, 1 can go to any $i \in \{1, \ldots, n\}$.
- Then 2 is at $i-1$ or $i+1$

Reflect about

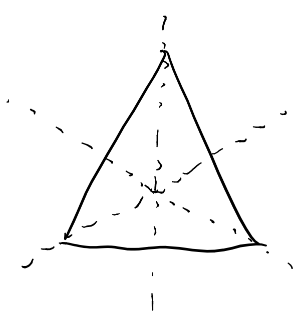


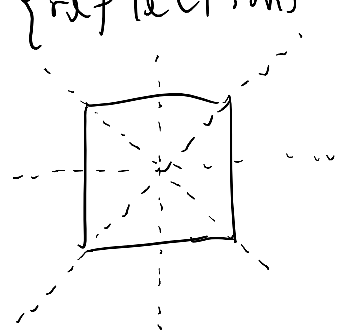
to realize the other pos'n for 2.

Thus $|D_{2n}| = 2n$. $\square$

Note $D_{2n} = \{\text{rotations thru } 2\pi i/n \text{ radians} \mid i = 0, 1, \ldots, n-1\}$

$\cup \{\text{reflections thru } n \text{ lines of symmetry}\}$





Standard notation

r = ccw rotation thru $2\pi/n$

s = reflection thru line joining 1 & center

Note ① $1, r, r^2, r^3, \ldots, r^{n-1}$ distinct

$r^n = 1 \implies |r| = n$.

② $|s| = 2$.

③ $s \neq r^i \;\forall i$

④ $sr^i \neq sr^j$ for $0 \leq i, j \leq n-1,\ i \neq j$

⑤ $rs = sr^{-1}$

⑥ Also $r^i s = sr^{-i}$.

$$D_{2n} = \{1, r, r^2, \ldots, r^{n-1}, s, sr, sr^2, \ldots, sr^{n-1}\}$$

Presentation:

$$D_{2n} = \langle \underbrace{r, s}_{\text{generators}} \mid \underbrace{r^n = s^2 = 1,\ rs = sr^{-1}}_{\text{relations.}} \rangle$$