Lecture 19

Wednesday, February 25, 2015 10:05 AM

Groups of order pq

$|G| = pq$

p, q primes, $p < q$.

$P \in Syl_p(G)$, $Q \in Syl_q(G)$.

By Sylow (3), $n_q \mid p$ & $n_q \equiv 1 \pmod q$

$n_q = 1 + kq$, $k \in \mathbb{N}$.

Thus $n_q = 1$ in order that $1+kq \mid p$.

$\Rightarrow Q \trianglelefteq G$.

What about n_p? $n_p \mid q \Longrightarrow n_p = 1$ or q.

If $p \nmid q-1$, then we must have $n_p = 1 \Rightarrow P \trianglelefteq G$.

Let $P = \langle x \rangle$, $Q = \langle y \rangle$. If $P \trianglelefteq G$, then

since $G/C_G(P) \cong$ subgp of $\underbrace{Aut(\mathbb{Z}_p)}_{\text{order } p-1}$

By a divisibility argument $[G : C_G(P)] = 1$

$\Rightarrow C_G(P) = G$. Then $x \in P \leq Z(G)$

so x, y commute thus $|xy| = pq \Rightarrow G = \langle xy \rangle \cong \mathbb{Z}_{pq}$.

What if $p \mid q-1$? Later: $\exists!$ nonabelian gp of order pq ($n_p = q$). [semi-direct products]

Suppose $|G| = 12$. Then either G has a normal Sylow 3-subgp or $G \cong A_4$ (in which case G has a normal Sylow 2-subgp $\cong \mathbb{Z}_2 \times \mathbb{Z}_2$).

Pf Assume $|G| = 12$ & $n_3 > 1$.

By Sylow 3, $n_3 \equiv 1 \pmod 3$, $n_3 \mid 4 \implies n_3 = 4$.

Thus G contains $\underline{8}$ elements of order 3. Let $P \in Syl_3(G)$.

Note $4 = n_3 = [G : N_G(P)] \implies N_G(P) = P$.

$G \underset{\text{conj}}{\circlearrowright} Syl_3(G) \rightsquigarrow \varphi : G \longrightarrow S_{Syl_3(G)} \cong S_4$

Then $K = \ker(\varphi) = \{g \in G \mid g \text{ normalizes each Sylow 3-subgp}\}$

$\leq N_G(P) = P$

$\implies K \leq \bigcap_{P \in Syl_3(G)} P = 1 \implies K = 1 \implies \varphi$ injective.

Thus $G \cong \varphi(G) \leq S_4$.

G has 8 elts of order 3 & there exactly 8 elements of S_4 of order 3, each of which is an even permutation, i.e. $\in A_4$.

Thus $|\varphi(G) \cap A_4| \geq \underbrace{8}_{\text{order } 3} + \underbrace{1}_{id} = 9$

$|S_4| = 24$. $\{$divisors of 24 which are $\geq 9\}$ $= \{12, 24\}$.

Thus $|\varphi(G) \cap A_4| = 12 \implies \varphi(G) = A_4$

$\cong G$ $\cong$

$\square$

<u>Prop</u> $|G| = p^2 q$, p, q distinct primes then G has a normal Sylow subgp.

<u>Pf</u> <u>Case 1</u> $p > q$. Then $n_p \equiv 1 \pmod p$ & $n_p \mid q$

$\Rightarrow n_p = 1$, & G has a normal Sylow p-subgp.

<u>Case 2</u> $p < q$. If $n_q = 1$, we're done. Assume $n_q > 1$.

Then $n_q = 1 + kq$, some $k \in \mathbb{N}$ & $n_q \mid p^2$

$\Rightarrow n_q = p$ or p^2. But $n_q \neq p$ b/c $q > p$.

Thus $n_q = p^2$, $\underbrace{p^2 - 1}_{= (p-1)(p+1)} = kq \Rightarrow q \mid p-1$ or $q \mid p+1$.

$q \nmid p-1$ b/c $q > p$. So $q \mid p+1 \Rightarrow q = p+1$

$\Rightarrow p = 2$, $q = 3$ so $|G| = 12$. $\blacksquare$

<u>Defn</u> A subgp $H \leq G$ is <u>characteristic</u> in G if $\forall \sigma \in \mathrm{Aut}(G)$, $\sigma(H) = H$; write H char G.

Observations

1. H char $G \Rightarrow H \trianglelefteq G$

2. If H is the unique subgp of G of a given order then H char G.

3. K char H & $H \trianglelefteq G \Rightarrow K \trianglelefteq G$.

$\underbrace{conj_g \in Aut(G)}_{\text{has restriction}}$ in $Aut(H)$ b/c $H \trianglelefteq G$.

so $conj_g(K) = gKg^{-1} = K \quad \forall g \in G$.