Lecture 18

Tuesday, February 24, 2015 9:58 AM

Recall $P \in Syl_p(G)$, $\mathcal{S} = \{gPg^{-1}\} = \{P_1, \dots, P_r\}$.

$Q \leq_p G$, $Q \overset{\text{conj}}{\circlearrowright} \mathcal{S} \rightsquigarrow \mathcal{S} = \mathcal{O}_1 \sqcup \mathcal{O}_2 \sqcup \dots \sqcup \mathcal{O}_s$

for $\mathcal{O}_i$ the orbits of . $r = |\mathcal{S}| = |\mathcal{O}_1| + |\mathcal{O}_2| + \dots + |\mathcal{O}_s|$.

Prop $r \equiv 1 \pmod p$. $\square$ $P \in Syl_p(G)$.

Pf Sylow (2) Take $Q \leq_p G$. Suppose for contradiction

$Q \nleq gPg^{-1}$ $\forall g \in G$. Then $Q \nleq P_i$, $1 \leq i \leq r$

Thus $Q \cap P_i \lneq Q$, $1 \leq i \leq r$ so $|\mathcal{O}_i| = [Q : Q \cap P_i]$

> 1. Thus $p \mid |\mathcal{O}_i|$ $\forall i$. so $p \mid r$ $\cancel{\text{Q}}$ Prop.

If $P, Q \in Syl_p(G)$, then $Q \leq_p G$ so $\exists g \in G$ s.t.

$Q \leq gPg^{-1} \implies Q = gPg^{-1}$ b/c both of order p^α. $\square$

Pf Sylow (3) By Sylow (2), $\{gPg^{-1} \mid g \in G\} =$

$\{P_1, \dots, P_r\} = Syl_p(G)$. thus $r = n_p \equiv 1 \pmod p$.

$n_p = |\text{orbit of } P \text{ under conj by } G|$

$= [G : N_G(P)]$ by orbit-stabilizer. $\square$

<u>Cor</u> For $P \in Syl_p(G)$, TFAE:

① $\{P\} = Syl_p(G)$, i.e. $n_p = 1$

② $P \trianglelefteq G$

③ $\left(X \subseteq G,\ |x| = p^n\ \forall x \in X\right) \Rightarrow \langle X \rangle \leq_p G$.

<u>Pf</u> ① $\Rightarrow$ ②: $\forall g \in G$. $gPg^{-1} \in Syl_p(G) = \{P\}$ so $gPg^{-1} = P\ \forall g \in G \Rightarrow P \trianglelefteq G$.

② $\Rightarrow$ ①: $P, Q \in Syl_p(G)$. By Sylow ②, $Q = gPg^{-1} = P$. ($P \trianglelefteq G$)

① $\Rightarrow$ ③: For $x \in X$, $\langle x \rangle \leq_p G$. Thus by Sylow ②, $\exists g \in G$. $gPg^{-1} \geq \langle x \rangle$
In particular, $x \in gPg^{-1} = P \Rightarrow X \subseteq P$.
Thus $\langle X \rangle \leq P$. $\Rightarrow \langle X \rangle \leq_p P \leq_p G$.

$\langle x_1, x_2, x_3, \ldots \rangle$
$= \langle \{x_1, x_2, x_3, \ldots\} \rangle$

③ ⟹ ①: $X = \bigcup_{P \in Syl_p(G)} P$. Then $\forall x \in X$, $|x|$ is a p-power. Thus $\langle X \rangle \leq_p G$. Moreover, for $P \in Syl_p(G)$

$P \leq \langle X \rangle$ b/c $P \subseteq X$. By maximality of P amongst p-subgps, learn $P = \langle X \rangle$

$\Rightarrow Syl_p(G) = \{ \langle X \rangle \}$. $\square$

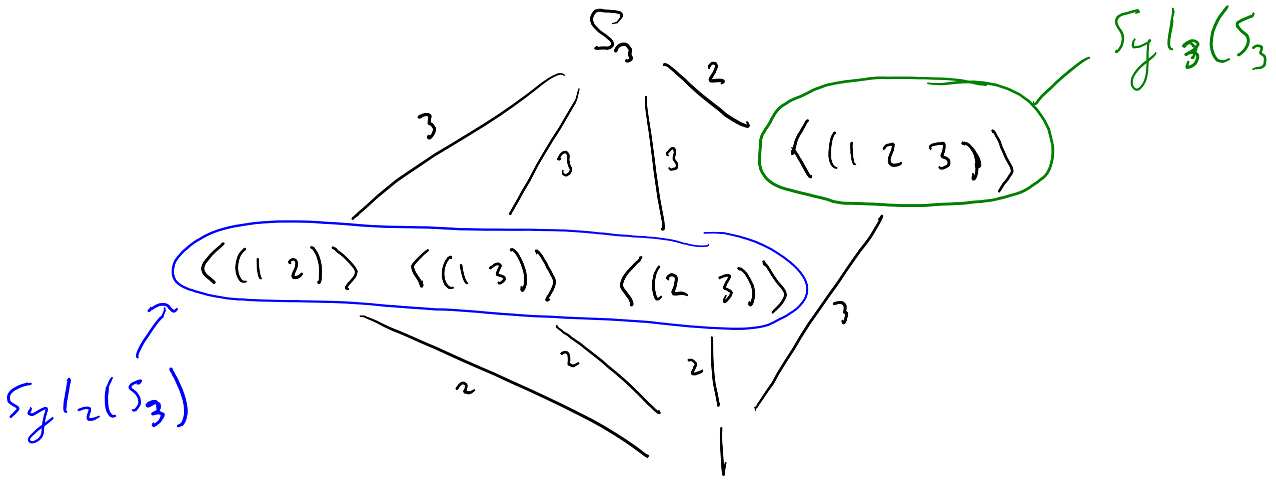
<u>e.g.</u> $|S_3| = 6 = 2 \cdot 3$

$Syl_3(S_3) = \{ \langle (1\ 2\ 3) \rangle \}$

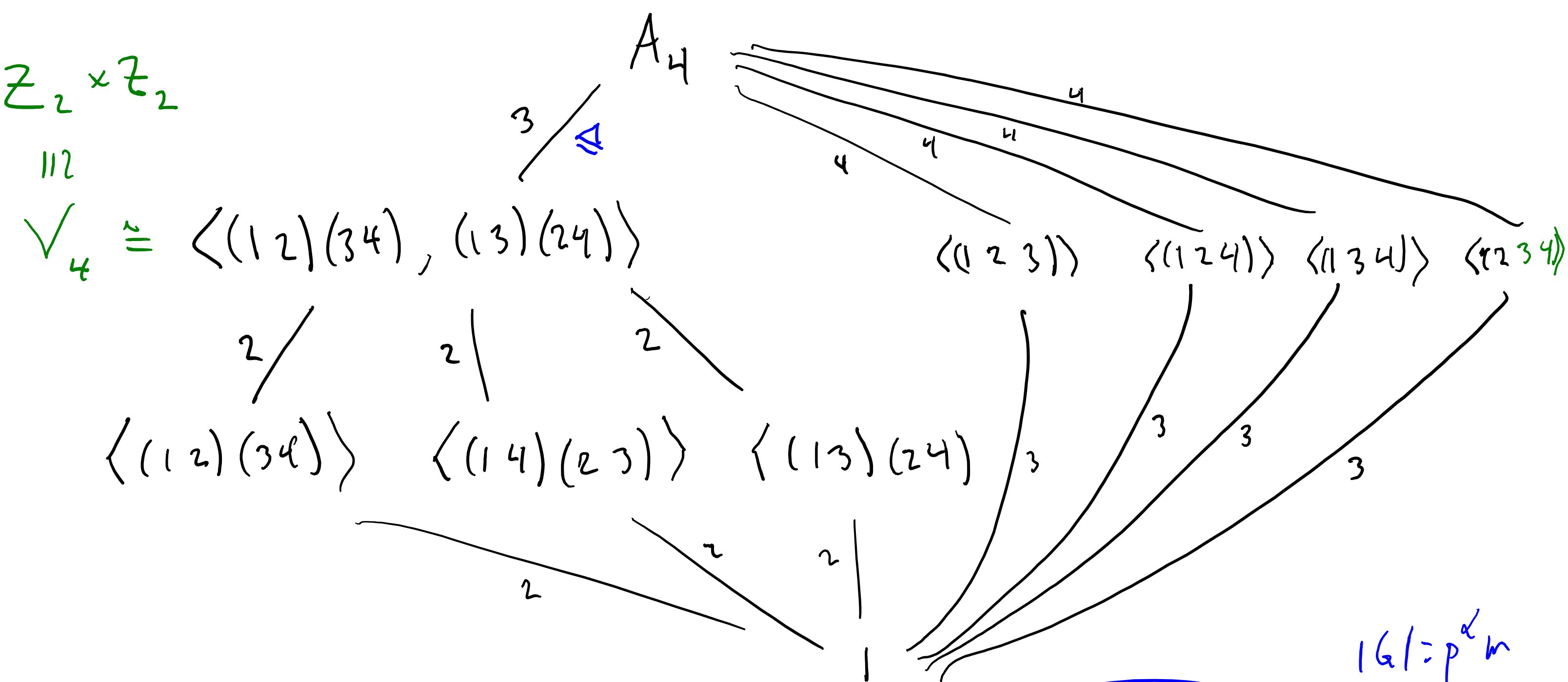
$Syl_2(S_3) = \{ \langle (1\ 2) \rangle, \langle (1\ 3) \rangle, \langle (2\ 3) \rangle \}$

$n_3 = 1 \equiv 1 \ (3)$ & $1 \mid 2$

$n_2 = 3 \equiv 1 \ (2)$ & $3 \mid 3$



e.g. A_4. $|A_4| = 12 = 2^2 \cdot 3$



$n_2 = 1 \equiv 1 \pmod 2$, $1 \mid 3$ ✓

$n_3 = 4 \equiv 1 \pmod 3$, $4 \mid 4$ ✓

$|G| = p^\alpha m$, $p \nmid m$

③ $n_p \equiv 1 \pmod p$

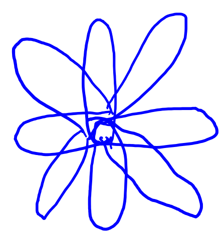
$n_p = [G : N_G(P)] \mid m$

Claim G is a gp of order 56.
Then G has a normal p-Sylow subgp for some prime $p \mid 56$.
In particular, G is not simple.

Pf $56 = 2^3 \cdot 7$ thus $n_7 = 1 + 7k$, some $k \in \mathbb{N}$ and $n_7 \mid 8$ by Sylow 3. Thus $n_7 = 1$ or 8.

If $n_7 = 1$ then the ! Sylow 7-subg is normal.
If $n_7 = 8$, there are 8 cyclic subgps of order 7 in G pairwise intersect in 1.
Thus there are $\underline{48}$ order 7 elements of G

$$\underbrace{48 = 6}_{\text{\# order 7 elts in } \mathbb{Z}_7} \cdot \underbrace{8}_{\text{\# of such gps w/ pairwise triv intersection}}$$



There are only 8 elements left; they form the unique Sylow 2-subgp. $\square$