Lecture 16

Friday, February 20, 2015 10:12 AM

$\mathrm{Aut}(G) = \{$ self-isos $G \to G\}$ gp under $\circ$

<u>Prop</u> If $H \trianglelefteq G$, then G acts on H via conjugation:

$g \in G$, $h \in H$, $g * h = ghg^{-1} \in H$ by normality.

G acts (by conjugation) is via automorphisms:

$$\mathrm{conj}_g : H \longrightarrow H \qquad \in \mathrm{Aut}(H)$$
$$h \longmapsto ghg^{-1}$$

Thus we get $\psi : G \longrightarrow \mathrm{Aut}(H)$

$$g \longmapsto \mathrm{conj}_g$$

has kernel $C_G(H)$. Thus by 1st iso thm

$$G / C_G(H) \cong \mathrm{im}(\psi) \leq \mathrm{Aut}(H).$$

<u>Pf</u> $\mathrm{conj}_g(hk) = ghkg^{-1} = (ghg^{-1})(gkg^{-1})$ with $\underbrace{g^{-1}g}_{1}$

$= \mathrm{conj}_g(h)\,\mathrm{conj}_g(k)$.

$\mathrm{conj}_{g^{-1}}$ is a 2-sided inverse of $\mathrm{conj}_g \Rightarrow \mathrm{conj}_g \in \mathrm{Aut}(H)$

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$$\ker(\Psi) = \{g \in G \mid \text{conj}_g = id\}$$
$$= \{g \in G \mid ghg^{-1} = h \ \forall h \in H\}$$
$$= C_G(H). \quad \square$$

<u>Cor</u> If $K \leq G$, then $K \cong gKg^{-1}$ $\forall g \in G$.

<u>Cor</u> If $H \leq G$, then $N_G(H)/C_G(H) \cong$ subgp of $Aut(H)$

$\underbrace{G/Z(G)}_{\substack{\text{Take } H = G \\ C_G(G) = Z(G)}} \cong$ subgp of $Aut(G)$. $\square$

<u>Defn</u> $Inn(G)$ = inner automorphisms of G
= {conjugation autos of G}

<u>Note</u> $Inn(G) \cong G/Z(G)$.
$$\text{conj}_g \longmapsto gZ(G)$$

e.g. · G abelian $\Rightarrow$ $\mathrm{Inn}(G) = 1$

· $G = Q_8$. $Z(Q_8) = \langle -1 \rangle = \{\pm 1\}$

$Q_8/\{\pm 1\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathrm{Inn}(Q_8)$

· If $H \le G$ & $H \cong \mathbb{Z}_2$, then $\mathrm{Aut}(H) = 1$

$\cong N_G(H)/C_G(H) \Rightarrow N_G(H) = C_G(H)$.

· If $H \trianglelefteq G$, $H \cong \mathbb{Z}_2$, then $N_G(H) = G$

so $C_G(H) = N_G(H) = G$ thus

$H \le Z(G)$.

Aside $\varphi \in \mathrm{Aut}(G)$ is inner if $\exists g \in G$ s.t.

$\varphi = \mathrm{conj}_g$.

G finite $\Rightarrow$ $|\mathrm{Inn}(G)| = [G : Z(G)]$

Prop $\mathrm{Aut}(Z_n) \cong (\mathbb{Z}/n\mathbb{Z})^\times$

In particular, $\mathrm{Aut}(Z_n)$ is abelian of order $\varphi(n)$, φ Euler's totient function:

$$\varphi(n) = \Big|\underbrace{\{a \in \mathbb{Z}^+ \mid (a,n)=1,\ 1 \le a \le n\}}_{\text{"totatives" of } n}\Big|.$$

Pf Suppose $Z_n = \langle x \rangle$. If $\psi \in \mathrm{Aut}(Z_n)$, then $\psi(x) = x^a$ for some $a \in \mathbb{Z}$. Moreover, the integer a uniquely determines ψ: $\psi(x^b) = (\psi(x))^b = (x^a)^b = x^{ab}$.

Let ψ_a be a hom taking x to x^a.

Since $|x| = n$, $\psi_a = \psi_b$ whenever $a \equiv b \ (n)$.

ψ_a an automorphism $\Rightarrow$ x^a has the same order as x

$$|x^a| = \frac{n}{(n,a)}$$

$$\Rightarrow (n,a) = 1.$$

Get

$$\Psi : \mathrm{Aut}(\mathbb{Z}_n) \longrightarrow (\mathbb{Z}/n\mathbb{Z})^\times \qquad \text{a surjective function}$$

$$\psi_a \longmapsto [a] = a + n\mathbb{Z}$$

Is Ψ a homomorphism?

$$\Psi(\psi_a \circ \psi_b) \overset{?}{=} \Psi(\psi_a) \cdot \Psi(\psi_b)$$

$$\Psi(\psi_a \circ \psi_b) = \Psi(\psi_{ab}) = ab + n\mathbb{Z}, \qquad \Psi(\psi_a) \cdot \Psi(\psi_b) = (a+n\mathbb{Z}) \cdot (b+n\mathbb{Z}) = ab + n\mathbb{Z} \ \checkmark$$

$$\underbrace{\psi_a \circ \psi_b}_{} \overset{?}{=} \psi_{ab}$$

$$\begin{tikzcd} x \arrow[r, mapsto, "\psi_b"'] \arrow[rrr, bend right, mapsto, "\psi_{ab}"'] & x^b \arrow[r, mapsto, "\psi_a"'] & (x^b)^a = & x^{ab} \end{tikzcd}$$

Yes!

Is Ψ injective?

$$\psi_a = \psi_b \iff x^a = x^b$$

$$\iff a \equiv b \ (n)$$

$$\iff \underbrace{a + n\mathbb{Z}}_{=\Psi(\psi_a)} = \underbrace{b + n\mathbb{Z}}_{=\Psi(\psi_b)}.$$