Lecture 15

Wednesday, February 18, 2015   10:01 AM

$G \circlearrowright G$ left mult $G \circlearrowright X$ Problem 8!

$$\begin{tikzcd}
G \arrow[r, "F"] \arrow[d, bend right] & X \\
G/H \arrow[ur, dashed, "\bar F ?"'] &
\end{tikzcd}$$

s.t. $g \cdot a \longmapsto g \cdot F(a) = F(g \cdot a)$

Claim $\bar F$ exists iff $\boxed{H \le \ker(G \circlearrowright X)}$

$\forall h \in H.\ \forall x \in X$

$$\bar F(gH) = F(g)$$
$$\parallel \quad \text{whenever } gH = g'H$$
$$\bar F(g'H) = F(g') \quad \text{i.e. } g = g'h,\ h \in H$$

So need $F(g'h) = F(g')\ \forall g' \in G,\ h \in H.$

$$\begin{array}{ccc} F(g'h) & = & F(g') \\ \parallel & & \parallel \\ g' \cdot F(h) & & g' \cdot F(1) \end{array}$$

$F(h) = F(1)$
$\forall h \in H$

Conjugacy in S_n : <u>Prop</u> $\sigma \in S_n$ has cycle decomp

$$\sigma = (a_1\ a_2 \cdots a_{k_1})(b_1\ b_2 \cdots b_{k_2})\cdots$$

$\Updownarrow$

"change of basis"

Then if $\tau \in S_n$

$$\tau\sigma\tau^{-1} = (\tau(a_1)\ \tau(a_2) \cdots \tau(a_{k_1}))$$

$$(\tau(b_1)\ \tau(b_2) \cdots \tau(b_{k_2}))$$

$$\vdots$$

<u>Pf</u> If $\sigma(i) = j$, then $\tau\sigma\tau^{-1}(\tau(i)) = \tau\sigma(i)$

$= \tau(j)$ b/c $\sigma(i) = j$. $\square$

<u>Defn</u> If $\sigma \in S_n$ is the product of disjoint cycles (including 1-cycles potentially) of length $n_1, n_2, \ldots, n_r$ w/ $n_1 \leq n_2 \leq \cdots \leq n_r$, then the integers $n_1, \ldots, n_r$ are called the <u>cycle type</u> of σ.

<u>Note</u> $n_1, \ldots, n_r$ give us a <u>partition</u> of n: positive integers which sum to n.

<u>e.g.</u> $\sigma = (1\ 2\ 3)(5\ 6) \in S_6$ has cycle type
$= (4)(5 6)(1 2 3)$ 1, 2, 3.

Prop Two elts of S_n are conjugate iff they have the same cycle type.

Pf $\Rightarrow$: $\sigma' = \tau \sigma \tau^{-1}$ then σ, σ' have same cycle type by the previous prop.

$\Leftarrow$: If σ_1, σ_2 have the same cycle type, write them in nondecreasing cycle length order. Get 2 lists of ~~#~~s $1, \ldots, n$. Define

τ (i-th integer listed for σ_1) = i-th integer listed for σ_2

Then $\tau \in S_n$ & $\tau \sigma_1 \tau^{-1} = \sigma_2$. $\square$

$$\begin{array}{l} \sigma_1 = (4)(5\ 6)(1\ 2\ 3) \\ \quad\quad\ \ \downarrow\ \downarrow\ \downarrow\ \ \downarrow\ \downarrow\ \downarrow\ \tau \\ \sigma_2 = (3)(1\ 2)(4\ 6\ 5) \end{array}$$

$$\tau \sigma_1 \tau^{-1} = (\tau(4))(\tau(5)\ \tau(6))(\tau(1)\ \tau(2)\ \tau(3))$$
$$= (3)(1\ 2)(4\ 6\ 5) = \sigma_2 \checkmark$$

e.g. Representatives for conj classes in S_4 are in bij corr w/ partitions of 4

Partition of $n=4$	conj class rep of that cycle type
$1+1+1+1$	1
$1+1+2$	$(2\ 3)$
$1+3$	$(2\ 3\ 4)$
4	$(1\ 2\ 3\ 4)$
$2+2$	$(1\ 4)(2\ 3)$

Observe For $H \trianglelefteq G$, $\mathcal{K}$ conj class in G, then either $\mathcal{K} \subseteq H$ or $\mathcal{K} \cap H = \emptyset$.
(b/c $ghg^{-1} \in H$ $\forall h \in H, g \in G$)

Thm A_5 is a simple group.

Pf There are 5 conjugacy classes in A_5 w/ representatives & sizes:

rep	1	$(1\ 2\ 3\ 4\ 5)$	$(2\ 1\ 3\ 4\ 5)$	$(1\ 2)(3\ 4)$	$(1\ 2\ 3)$
size	1	12	12	15	20

rep	1	(1 2 3 4 5)	(2 1 3 4 5)	(12)(34)	(1 2 3)
size	1	12	12	15	20

Suppose $H \trianglelefteq A_5$. Then H is a union of conjugacy classes one which is $\{1\}$.

Thus $|H|$ is 1+ a sum of #s 12, 12, 15, 20.

All possible sums do <u>not</u> divide 60 except for 1 & 60. By Lagrange's theorem,

$$|H| \,\big|\, |A_5| = 60 \text{ so } H = 1 \text{ or } A_5. \qquad \square$$

<u>Automorphisms</u> An <u>automorphism</u> of a gp G is is an isomorphism $G \longrightarrow G$; the set of these is $\mathrm{Aut}(G)$ & forms a gp under compositions.

<u>Note</u> $\mathrm{Aut}(G) \leq S_G$.

- identity in $\mathrm{Aut}(G)$ is id_G.
- $f : G \longrightarrow G \in \mathrm{Aut}(G)$ has an inverse as a fn. Can check f^{-1} is also a homomorphism.

$$f \circ f^{-1} = f^{-1} \circ f = id_G$$

- $f, g \in Aut(G)$, $f \circ g$ is hom & bijective.