Lecture 14

Tuesday, February 17, 2015 10:03 AM

Pf ① $G \cdot H = \{gH \mid g \in G\} = G/H$. ✓

② $G_H = \{g \mid gH = H\} = H$. ✓

③ $\ker(\pi_H) = \{g \mid gxH = xH \ \forall x \in G\}$

$= \{g \mid x^{-1}gxH = H \ \forall x \in G\}$

$= \{g \mid x^{-1}gx \in H \ \forall x \in G\}$

$= \{g \mid g \in xHx^{-1} \ \forall x \in G\}$

$= \bigcap_{x \in G} xHx^{-1}$. $\square$

Cor G finite gp of order n & let p be the smallest prime dividing n. If $H \leq G$ & $[G:H] = p$, then $H \trianglelefteq G$.

Pf $K = \ker(\pi_H) = \bigcap_{x \in G} xHx^{-1} \leq H$. Let $[H:K] = k$.

Then $[G:K] = [G:H][H:K] = pk$.

$G/K \cong$ subgp of $S_p = \mathrm{im}(\pi_H : G \longrightarrow S_{G/H})$

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Thus $pk \mid |S_p| = p! \Rightarrow k \mid (p-1)!$

$\uparrow$ prime factors $< p$.

Since $k \mid |G|$ & p smallest prime factor of $|G|$,

$k = 1$. $H = K \Rightarrow H \trianglelefteq G$. $\square$

Conjugation $\quad G \circlearrowright G$ via conjugation: $g \in G$, $a \in G$

$$\underset{\substack{\uparrow \\ \text{conj. action}}}{g * a} = \underbrace{gag^{-1}}_{\text{product in } G}$$

Defn $a, b \in G$ are conjugate (or in the same conjugacy class) if in the same orbit under conj.

i.e. $\exists g \in G$ s.t. $a = g * b = gbg^{-1}$.

$G \circlearrowright \mathcal{P}(G)$: $S \subseteq G$, $g * S = gSg^{-1} = \{gsg^{-1} \mid s \in S\}$.

Recall: $G_S = N_G(S)$

If $S = \{s\}$, $G_{\{s\}} = C_G(s) = N_G(\{s\})$.

Prop ① $\left|\{\text{conjugates of } S \subseteq G\}\right| = [G : N_G(S)]$

② $\left|\{\text{conjugates } s \in G\}\right| = [G : C_G(s)]$.

Pf Orbit-stabilizer: $|Gx| = |G/G_x|$.

① $|G * S| = |G/G_S| = [G : N_G(S)]$.

② similar. $\square$

Thm [Class equation] Let G be a finite gp, $g_1, g_2, \dots, g_r$ be representatives of the distinct conjugacy classes of G not contained in $Z(G)$. Then

$$\boxed{|G| = |Z(G)| + \sum_{i=1}^{r} [G : C_G(g_i)]}.$$

Pf Conjugacy classes partition G.

$Z(G)$ is the collection of elts in conj classes of size 1.

$|G * g_i| = [G : C_G(g_i)]$ by orbit-stabilizer.

$\square$

e.g. ① G abelian. Then $Z(G) = G$ & class eq'n reads $|G| = |Z(G)|$.

② $G = Q_8$. $\langle i \rangle \leq C_{Q_8}(i) \leq Q_8$.

$i \notin Z(Q_8)$ b/c $ij = k$, $ji = -k$.

and $[Q_8 : \langle i \rangle] = 2 \implies C_{Q_8}(i) = \langle i \rangle$.

Conjugates of i are i and $-i = kik^{-1}$.

Get conj classes $\underbrace{\{1\}, \{-1\}}$, $\underbrace{\{\pm i\}, \{\pm j\}, \{\pm k\}}$.

Indeed $8 = 2 + (2 + 2 + 2)$.

Thm p prime, P finite gp of order p^α, $\alpha \geq 1$. Then P has nontrivial center $Z(P)$.

Pf $$p^\alpha = |Z(P)| + \sum_{i=1}^{r} \underbrace{[P : C_P(g_i)]}_{\neq 1 \text{ so } p \,\mid\, [P : C_P(g_i)]}.$$

$Z(P) = 1$ would give $p^\alpha = 1 + p(\sim)$.

In fact $p \mid |Z(P)|$. $\square$

$$[P : C_P(g_i)] = |P/C_P(g_i)| = p^\alpha / |C_P(g_i)| = p^\beta, \quad 1 < \beta$$

<u>Cor</u> If $|P| = p^2$, p prime, then

$$P \cong \mathbb{Z}_{p^2} \quad \text{or} \quad \mathbb{Z}_p \times \mathbb{Z}_p .$$

In particular, P is abelian.

<u>Lemma</u> If $G/Z(G)$ is cyclic, then G abelian.

<u>Pf Cor</u> If $Z(P) \neq 1$, then $P/Z(P)$ has order 1 or p and thus cyclic. By the lemma, P is abelian. If P has an elt x of order p^2, then $P \cong \mathbb{Z}_{p^2}$ $= \langle x \rangle$. Otherwise, every $x \in P \setminus \{1\}$ has order p. Take $x \in P \setminus \{1\}$, $y \in P \setminus \langle x \rangle$.

$$|\langle x, y \rangle| > |\langle x \rangle| = p \quad \text{but} \quad \langle x, y \rangle \leq P \quad \text{so}$$

$$|\langle x, y \rangle| = p^2 \implies \langle x, y \rangle = P .$$

$$\begin{array}{ccc} (x^a, y^b) & \langle x \rangle \times \langle y \rangle & \cong \mathbb{Z}_p \times \mathbb{Z}_p \\ \big\downarrow & \big\downarrow \cong & \\ x^a y^b & P & . \end{array}$$

$\square$ / lemma

Pf lemma. Suppose $G/Z(G) = \langle xZ(G) \rangle$.

Then every elt of G is of the form $x^a z$, $z \in Z(G)$, $a \in \mathbb{Z}$. $a, b \in \mathbb{Z}$, $z, w \in Z(G)$

Then $x^a z\, x^b w = x^a x^b z w$

$= x^b x^a z w$

$= x^b w\, x^a z$. $\square$