# Lecture 13

Monday, February 16, 2015 9:31 AM

Recall sign hom $\varepsilon : S_n \longrightarrow \{\pm 1\}$.

$\varepsilon(\text{transposition}) = -1$

$\varepsilon(\sigma) = -1 \iff \sigma$ is a product of an odd # of transpositions

$\iff$ # of cycles of even length in $\sigma$'s cycle decomp is odd,

(b/c $(a_1\ a_2 \cdots a_m) = (a_1\ a_m)(a_1\ a_{m-1}) \cdots (a_1\ a_2)$.

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<u>Group Actions</u> $G \circlearrowright A$ when $G \times A \longrightarrow A$

$(g, a) \mapsto g \cdot a$

satisfies ① $1 \cdot a = a \ \forall a \in A$

② $g \cdot (h \cdot a) = (gh) \cdot a \ \forall g, h \in G,\ a \in A$.

For each $g \in G$ get $\sigma_g : A \to A$, $a \mapsto g \cdot a$

gives the permutation representation

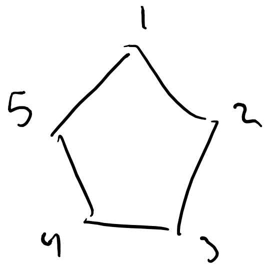
$$\varphi_{G \circlearrowright A} : G \longrightarrow S_A, \quad g \longmapsto \sigma_g$$

e.g. (1) $S_n \circlearrowright \{1, 2, \dots, n\} = \underline{n}$

$$S_n = \{\sigma : \underline{n} \longrightarrow \underline{n} \mid \sigma \text{ bij}\}$$

$$\sigma \cdot i = \sigma(i)$$

(2) $D_{2n} \circlearrowright$ {vertices of a regular $n$-gon}

$D_{10} \circlearrowright$ 


<u>Defn</u> • $\ker(G \circlearrowright A) = \ker(\varphi_{G \circlearrowright A}) \trianglelefteq G$

$$\ker(G \circlearrowright A) = \{g \in G \mid g \cdot a = a \ \forall a \in A\}$$

• for $a \in A$, $G_a = \{g \in G \mid g \cdot a = a\}$

$=$ <u>stabilizer</u> (or <u>isotropy</u>) of $a$.

• $G \circlearrowright A$ is <u>faithful</u> if $\ker(G \circlearrowright A) = 1$.

$K = \ker(G \circlearrowright A)$

$g, h$ act in the same way on $A \iff g \cdot K = h \cdot K$
(i.e. $\sigma_g = \sigma_h$)

so data of $G \circlearrowright A$ is the same as the faithful action of $G/K \circlearrowright A$.

$$G \xrightarrow{\varphi_{G \circlearrowright A}} S_A$$

$1^{st}$ iso thm tells us that $G/K \cong$ subgp of $S_A$
$\shortparallel$
$\mathrm{im}(\varphi_{G \circlearrowright A})$.

Prop $G \circlearrowright A$ induces an equivalence rel'n

$$a \sim b \iff a = g \cdot b \text{ for some } g \in G.$$

Pf $a = 1 \cdot a$ so $a \sim a$. If $a \sim b$ via $a = gb$, then $b = g^{-1} \cdot a$ so $b \sim a$. If $a \sim b$ $(a = gb)$ & $b \sim c$ $(b = hc)$ then

$$a = g(hc) = (gh)c \Rightarrow a \sim c. \quad \blacksquare$$

Thus $A$ is partitioned into equivalence classes: the orbits of $G \circlearrowright A$, $G \cdot a = \{g \cdot a \mid g \in G\}$, $a \in A$.

Orbit-stabilizer theorem $G \circlearrowright A$, $a \in A$

$$G/G_a \longrightarrow G\cdot a \quad \text{is a well-defined bijection.}$$
$$gG_a \longmapsto g\cdot a$$

Pf [HW]

For $H \leq G$, the cosets $G/H = \{gH \mid g \in G\}$ has a natural left $G$-action $G \circlearrowright G/H$:

$$g \cdot xH = (gx)\cdot H$$

Special property: transitive

$G \circlearrowright A$ is transitive if $G\cdot a = A$ for one (or any) $a \in A$.

$G \cdot H = G/H$ so $G \circlearrowright G/H$ is transitive.

Defn Suppose $G \circlearrowright A$, $G \circlearrowright B$. A $G$-equivariant map $A \longrightarrow B$ is a fn $f: A \longrightarrow B$ s.t.
$\forall g \in G$, $a \in A$. $f(g\cdot a) = g\cdot f(a)$.

<u>Application</u> Cycle decompositions

$\sigma \in S_n$, $G = \langle \sigma \rangle \circlearrowright \{1, 2, \dots, n\} = \underline{n}$.

$\underline{n} = \coprod$ $G$-orbits of $G \circlearrowright \underline{n}$.

Let $\mathcal{O} = G \cdot x$ for some $x \in \underline{n}$ (note $x = 1 \cdot x \in \mathcal{O}$).

OST:
$$\mathcal{O} \xrightarrow{\sim} G/G_x \qquad (G_x = \{g \in G \mid gx = x\})$$
$$\sigma^i x \longmapsto \sigma^i G_x$$

$G$ is cyclic (thus abelian) so $G_x \trianglelefteq G$

& $G/G_x$ is cyclic of order $d$ where

$d = \min\{k \mid \sigma^k \in G_x\}$.

$\mathcal{O} = G \cdot x = \{x, \sigma x, \sigma^2 x, \dots, \sigma^{d-1} x\}$.

Thus $\sigma$ acts as a length $d$ cycle on orbits which have size $d$. $\rightsquigarrow$ cycle decomp'n of $\sigma$.

If we had chosen $\sigma^i x$ instead of $x$, we would have gotten $(\sigma^i x \ \sigma^{i+1} x \ \cdots \ \sigma^{d-1} x \ x \ \sigma x \ \cdots \ \sigma^{i-1} x)$

So up to reordering cycles and cyclically permuting how we write down each cycle, we have uniqueness $\square$

So up to reordering [illegible] and [illegible] permuting
how we write down each cycle, we have uniqueness ∎

Let's return $G \circlearrowright G/H$, $H \leq G$.

<u>Thm</u> Let $\pi_H = \varphi_{G \circlearrowright G/H}$. Then

① $G$ acts transitively on $G/H$.

② $G_H = H$

③ $\underbrace{\ker(\pi_H) = \bigcap_{x \in G} \underbrace{xHx^{-1}}_{\{xhx^{-1} \mid h \in H\}}}_{\text{largest normal subgp of } G \text{ containing } H.}$