Lecture 12

Friday, February 13, 2015 10:01 AM

Isomorphism theorems

① $\varphi: G \longrightarrow H$, $G/\ker(\varphi) \cong \operatorname{im}(\varphi)$

② $A, B \leq G$, $A \leq N_G(B) \Rightarrow AB \leq G$, $B \trianglelefteq AB$, $AB/B \cong A/A\cap B$

③ $H, K \trianglelefteq G$, $H \leq K \Rightarrow (G/H)/(K/H) \cong G/K$

④ Says what subgroups of G/N look like: $\{H/N \mid N \leq H \leq G\}$.

Simple group: G is <u>simple</u> if the only normal subgps of G are 1 and G.

Prop Suppose $N \trianglelefteq G$, N is solvable, G/N is solvable. Then G is solvable.

Pf Take $1 = N_0 \trianglelefteq N_1 \trianglelefteq N_2 \trianglelefteq \cdots \trianglelefteq N_n = N$ w/ N_{i+1}/N_i abelian. Sim,

$$1 = \overline{G}_0 \trianglelefteq \overline{G}_1 \trianglelefteq \cdots \trianglelefteq \overline{G}_m = G/N$$

w/ $\overline{G}_{i+1}/\overline{G}_i$ abelian. By 4th iso thm,

$\exists\, G_i \leq G$ w/ $N \leq G_i$ and $G_i/N = \overline{G}_i$.

Moreover $G_i \trianglelefteq G_{i+1}$.

Thus $1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_n = N = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_m = G$

Moreover, $$G_{i+1}/G_i \underset{\text{3rd iso thm}}{\cong} \left(G_{i+1}/N\right) \big/ \left(G_i/N\right) = \overline{G_{i+1}}/\overline{G_i}$$

which is abelian. ∎

The sign homomorphism & alternating group.

<u>Defn</u> A <u>transposition</u> in S_n is a length 2 cycle.

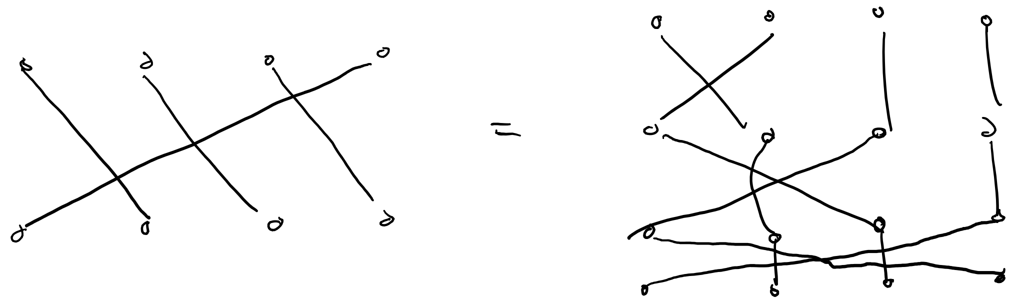
<u>e.g.</u> $(3 \; 5) \in S_6$

<u>Prop</u> Let $T = \{\text{transpositions} \in S_n\}$. Then

$$\langle T \rangle = S_n.$$

<u>Pf</u> Permutations have cycle decomps, so suffices to write $(a_1 \; a_2 \; \cdots \; a_m)$ as a product of transp'ns.

$$= (a_1 \; a_m)(a_1 \; a_{m-1}) \cdots (a_1 \; a_2).$$



∎

Action $S_n \circlearrowright \mathbb{Z}[x_1, x_2, \dots, x_n]$

$= \{$ polynomials in $x_1, \dots, x_n$ w/ coeffs in $\mathbb{Z}$ $\}$

defined by $\sigma f(x_1, x_2, \dots, x_n)$

$= f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$.

Let $\Delta = \prod_{1 \le i < j \le n} (x_i - x_j)$

e.g. $n = 3$: $\Delta = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$

$\sigma \in S_n$, $\sigma \cdot \Delta = \prod_{1 \le i < j \le n} (x_{\sigma(i)} - x_{\sigma(j)})$

$n = 3$:

$(1\ 2) \cdot \Delta = (x_2 - x_1)(x_2 - x_3)(x_1 - x_3)$

Note All $(x_{\sigma(i)} - x_{\sigma(j)})$ are of the form

$x_k - x_l$, $k < l$ or $x_l - x_k$, $k < l$

Thus $\sigma \cdot \Delta = \pm \Delta$.

<u>Defn</u> For $\sigma \in S_n$, $\varepsilon(\sigma) = \dfrac{\sigma \cdot \Delta}{\Delta} = \begin{cases} 1 & \text{if } \sigma\Delta = \Delta \\ -1 & \text{if } \sigma\Delta = -\Delta. \end{cases}$

$\varepsilon(\sigma)$ is the <u>sign</u> of σ.

If $\varepsilon(\sigma) = 1$, call σ <u>even</u>

$\varepsilon(\sigma) = -1$, call σ <u>odd</u>.

<u>Prop</u> $\varepsilon : S_n \longrightarrow \{\pm 1\}$ is a hom (where $\{\pm 1\}$ has mult'n as gp op).

<u>Pf</u> Want $\varepsilon(\sigma\tau) = \varepsilon(\sigma) \cdot \varepsilon(\tau)$.

$$(\sigma\tau) \cdot \Delta = \varepsilon(\sigma\tau)\, \Delta$$

$$\| $$

$$\varepsilon(\tau)\, \sigma \cdot \Delta = \varepsilon(\tau)\, \varepsilon(\sigma)\, \Delta = \varepsilon(\sigma)\, \varepsilon(\tau)\, \Delta$$

If $\varepsilon(\tau) = (-1)^k$ b/c τ swaps k of the factors, then

$(\sigma\tau)\Delta = \sigma \cdot (\tau \Delta)$ has k factors $x_{\sigma(j)} - x_{\sigma(i)}$ w/ $j > i$

Swap (change sign) of the $(x_{\sigma(j)} - x_{\sigma(i)})$'s w/ $j > i$ to get

$$\sigma\tau\Delta = \varepsilon(\tau)\,\varepsilon(\sigma)\,\Delta$$

Defn The alternating group (on n letters) is $A_n = \ker(\varepsilon : S_n \longrightarrow \{\pm 1\})$.

Facts
- A_n is simple (nonabelian) gp for $n \geq 5$.
- $[S_n : A_n] = 2$, in fact by 1st iso thm $S_n / A_n \cong \{\pm 1\}$ (at least if ε is surj)
- $|A_n| = \frac{1}{2} n!$
- $A_4 \cong$ gp of rigid transformations of tetrahedron

Q What does ε do to transpositions?

$$\varepsilon((1\ 2)) = -1$$

Let $\lambda = (1\ i)(2\ j)$. Then $\lambda (1\ 2) \lambda = (i\ j)$.

Friday, February 13, 2015 10:50 AM

Thus $\varepsilon((i\ j)) = \varepsilon(\lambda\ (1\ 2)\ \lambda)$

$= \varepsilon(\lambda)\ \varepsilon((1\ 2))\ \varepsilon(\lambda)$

$= \varepsilon(\lambda)^2 \cdot (-1)$

$= -1$

Prop $\varepsilon(\text{trans}) = -1$ & ε is surj. $\square$

Note A_n = gp of even permutations

$S_n \setminus A_n$ = set of odd permutations.