

Lecture 11

Composition series and the Holder program

Earlier we stated Cauchy's theorem: If G is a finite gp & p is a prime dividing $|G|$, then G has an element of order p .

We begin by proving an easier version:

Prop. If G is a finite abelian gp, then the conclusion of Cauchy's theorem holds.

Pf Assume p prime, $p \mid |G|$. We proceed by strong induction on $|G|$. The base case is $|G|=p$, in which case Lagrange's thm guarantees that for any $x \in G - \{1\}$, $p \mid |x|$. But then $|x|=p$, as desired.

Now assume $|G| > p$ and take $x \in G - \{1\}$. If $p \nmid |x|$, write $|x|=pn$. Then $|x^n|=p$, and we're done. We may thus assume $p \mid |x|$.

Let $N = \langle x \rangle$. Since G is abelian, $N \trianglelefteq G$.

$|G/N| = |G|/|N| < |G|$. Since $p \nmid |N|$, we have $p \mid |G/N|$. By induction, $\exists y \in G/N$ of order p . Now $y \notin N$ b/c $y^p \in N$ so $\langle y \rangle \neq \langle y^p \rangle \Rightarrow |y| < |y^p|$. Thus $p \mid |y|$, and the above argument produces an elt

of order p from y , completing our induction. \square

This proof exhibits a general technique in finite gphy:

Prove by induction on $|G|$ by thinking of G as being "built" from N and G/N .

Of course, such proofs depend on producing $N \trianglelefteq G$ w/ $N \neq 1, G$ — this isn't always possible! [But when G is abelian, it's easy as long as $G \cong \mathbb{Z}_p$.]

Defn A group G is called simple if $|G| > 1$ and the only normal subgps of G are 1 and G .

Note • Each \mathbb{Z}_p , p prime is simple.

- The first nonabelian simple gp has order 60. It is part of an infinite family of nonabelian simple gps.

- A 5,000-page theorem: There are 18 infinite families of simple groups and 26 simple gps not belonging to these families (the sporadic simple gps) such that every finite simple gp is isomorphic to one of these gps!

Idea Study finite gps by studying simple gps and how simple gps can be "put together" to get all finite gps.

Defn A sequence of subgps

$$1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_{k-1} \trianglelefteq N_k = G$$

is called a composition series if $N_i \trianglelefteq N_{i+1}$ and N_{i+1}/N_i is a simple gp for $0 \leq i < k$.
composition factors of G .



\trianglelefteq is not transitive!

In particular, each N_i only need be normal in N_{i+1} , and N_i probably isn't normal in G (unless $i = k-1$).

Jordan-Hölder theorem Let G be a finite gp.

Then ① G has a composition series.

② If $1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_r = G$ and

$$1 = M_0 \trianglelefteq M_1 \trianglelefteq \cdots \trianglelefteq M_s = G$$

are two composition series for G , then $r = s$ and there is a permutation σ of $\{1, 2, \dots, r\}$ s.t.

$$M_{\sigma(i)}/M_{\sigma(i)-1} \cong N_i/N_{i-1}$$

for $1 \leq i \leq r$.

Note composition factors : finite gps :: primes : integers.

Pf Deferred.

The following class of groups is important in Galois theory (algebraic study of roots of polynomials):

Defn A gp G is solvable if there is a chain of subgroups $1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_s = G$ s.t. G_{i+1}/G_i is abelian for $0 \leq i \leq s$.

Prop Suppose $N \trianglelefteq G$, N is solvable, and G/N is solvable/l. Then G is solvable.

Pf (by 3rd & 4th iso thms)

$$\text{Take } 1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_n = N$$

$$1 = \bar{G}_0 \trianglelefteq \bar{G}_1 \trianglelefteq \cdots \trianglelefteq \bar{G}_m = G/N$$

guaranteed by solvability. By 4th iso thm,

$$\exists G_i \trianglelefteq G \text{ s.t. } N \trianglelefteq G_i \text{ and } G_i/N = \bar{G}_i$$

and $G_i \trianglelefteq G_{i+1}$. By 3rd iso thm,

$$\bar{G}_{i+1}/\bar{G}_i = (G_{i+1}/N)/(G_i/N)$$

$\cong G_{i+1}/G_i$, so G_{i+1}/G_i is abelian as well.

Now $1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_n = N = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_m = G$ exhibits that G is solvable. \square